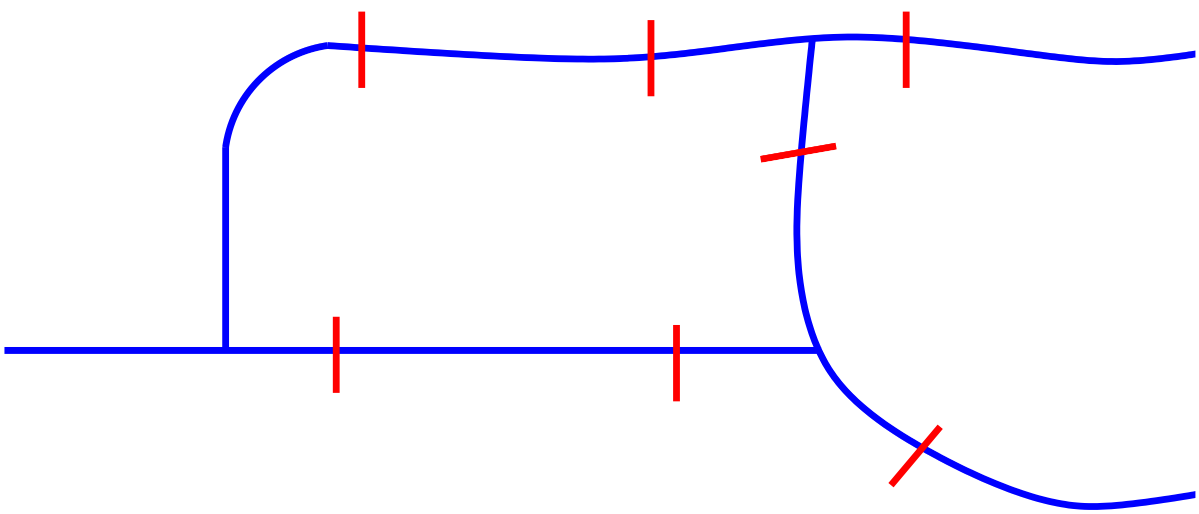
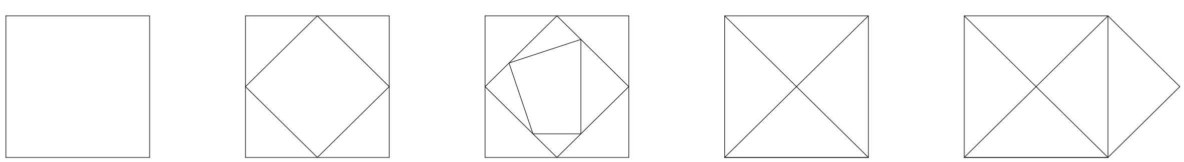
The Origins of Graph Theory

1 Two problems

The Königsberg bridge problem. The city of Königsberg lies along the Pregel River. The river has several branches, dividing the city into four districts, connected by seven[1] bridges, as in the figure shown below (rivers in blue, bridges in red). A longstanding puzzle for the residents of Königsberg was as follows: Is it possible to design a stroll around the city which would cross each of the seven bridges exactly once?



Unicursal drawings. Start by drawing a figure made out of line segments — a square, or a triangle, or a square inside another square, or any of the examples below.



Is it possible to draw that figure without ever (a) picking your pencil up off the paper or (b) retracing any segment you've already drawn? (A drawing with these properties is called "unicursal".)

In fact, these two problems are equivalent, albeit differently stated.

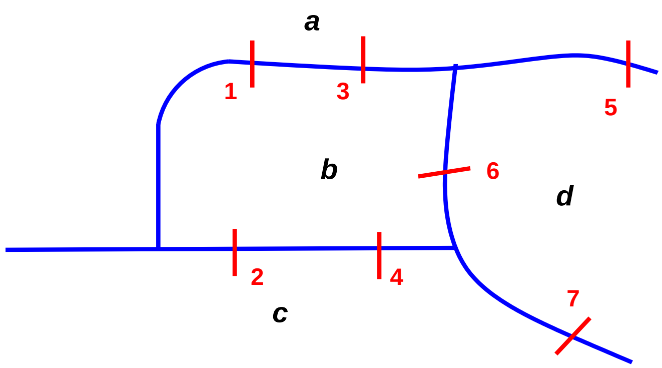
[1]Not any more. Two of the bridges were destroyed in World War II, two have been demolished, two more bridges have been built, and Königsberg is now Kaliningrad.

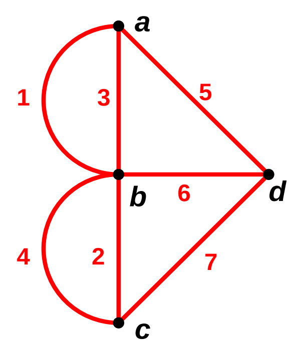
2 Graph theory

In 1736, the great Swiss mathematician Leonhard Euler solved the Königsberg bridge problem. Euler's key insight was that the islands and bridges could be modeled by a simple mathematical structure called a graph. Graph theory has since developed into an extremely beautiful and useful area of mathematics, with all kinds of theorems and applications.

Definition 1. A graph G consists of a set of vertices and a set of edges, where each edge connects two vertices.

For example, the map of Königsberg can be represented by a graph with four vertices a, b, c, d, representing the districts of the city, and seven edges $1, \ldots, 7$, representing the bridges. Edge #1 connects vertices a and b; edge #2 connects vertices b and c; etc.





It looks like there's a mistake in the figure — shouldn't edges 2 and 4 be interchanged? (After all, in the map, bridge 2 is west of bridge 4.) Actually, **it doesn't matter.** The definition of a graph has nothing to do with location; the only information the graph knows is the names of the vertices and edges and which is attached to which. So as long as both these edges connect the same pair of vertices (namely b and c), it doesn't matter how we draw them.

Graphs come up in all kinds of real-world situations:

- The Web can be thought of as a graph, where the vertices are webpages and the edges are links.
- The "Six Degrees of Separation" problem can be modeled by a graph whose vertices are people; two people are connected by an edge if they know each other.
- Family trees are graphs — vertices represent people and edges represent relationships such as marriage or parenthood. Similarly, so are the trees that evolutionary biologists use to model relationships between different species.
- The GPS device in your car uses graph theory to calculate the shortest driving route between two points. For example, edges are blocks and vertices are intersections.

Here are the graph-theoretic definitions we need to talk about the Königsberg bridge problem:

Definition 2. The degree of a vertex in a graph is the number of edges attached[2] to that vertex.

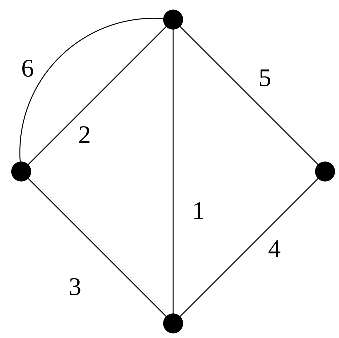
For example, vertex a has degree 3 (because it is attached to edges 1, 3, and 5) and vertex b has degree 5 (because it is attached to edges 1, 2, 3, 4, 6).

[2] There's one slight complication. It is permissible for an edge to connect a vertex to itself; such an edge is called a loop. Many of the graphs we want to study don't have any loops — for example, no bridge connects one of the districts of Königsberg to itself. However, if an edge connects a vertex to itself, we usually think of that edge as contributing 2, not 1, to the degree of the vertex. The reason for this will become clear later on.

Definition 3. An Eulerian path in a graph is a way to walk through the vertices of a graph, one edge at a time, so as to traverse every edge exactly once. (So an Eulerian path can be thought of as an order for the set of edges.)

An Eulerian circuit is an Eulerian path whose starting vertex is the same as its ending vertex.

For example, in the graph below, the sequence of edges 2, 6, 3, 1, 5, 4 forms an Eulerian path. It's not an Eulerian circuit because the starting and ending vertex are not the same.



The Königsberg bridge problem is simply this: If G is the graph whose vertices are regions ofKöniugsberg and whose edges are the bridges, then does G have an Eulerian path or an Eulerian circuit? Here is Euler's answer.

Theorem 4. *If G is a connected*[3] *graph, then:*

- *If G has no vertices of odd degree, then G has an Eulerian circuit.*
- *If G has 2 vertices of odd degree, then G has an Eulerian path but no Eulerian circuit.*
- *If G has 4 or more vertices of odd degree, then G has no Eulerian path (or Eulerian circuit).*

Here's part of Euler's reasoning. Suppose that a graph G has an Eulerian path P. If v is a vertex that is neither the first nor last vertex of P, then P must enter v exactly as many times as it leaves v. Since every edge incident to v is traversed exactly once, this means that the number of such edges — that is, the degree of P — must be even. Therefore, G has at most two vertices of odd degree, namely the first and last vertices of P. On the other hand, if P was actually an Eulerian circuit (not just an Eulerian path), then the first and last vertices are the same vertex x, and in fact x has even degree (because, again, P entered x exactly as many times as it left x). So in this case G has no vertices of odd degree.

This argument is not a complete proof; it is still necessary to show that if G has zero or two odd-degree vertices, then G really does have an Eulerian circuit or Eulerian path respectively. (Another way of saying this is that we have to rule out the possibility of a connected graph that happens to have, say, zero odd-degree vertices but happens to have no Eulerian circuit.) But this can be done.

There are a couple of obvious missing cases. What if G has one or three odd vertices? Investigating that is a homework problem.

By the way, remember that a loop contributes 2, not 1, to the degree of the vertex x to which it is attached. (See the footnote on the preceding page.) This rule makes sense in light of Euler's theorem. On the one hand, adding a loop at x doesn't affect the existence or non-existence of an Eulerian path or circuit — just insert the loop into an Euler path whenever you're standing at x. On the other hand, if the loop contributes 2 to the degree of its vertex, then ignoring loops doesn't affect the number of odd-degree vertices in the graph.

[3]This means that it is possible to walk from any vertex to any other by some sequence of edges.