First, I made a mistake in class when talking about the vector field

$$\mathbf{F}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^4}.$$

As pointed out by a couple of students, this field actually has negative divergence where it is defined (i.e., away from the origin). Algebraically, this is because

$$\begin{aligned}
(\nabla \cdot \mathbf{F})(x,y) &= \frac{\partial}{\partial x}\mathbf{F}_x + \frac{\partial}{\partial y}\mathbf{F}_y \\
&= \frac{(x^2+y^2)^4 - (x)(4(x^2+y^2)^3(2x)}{(x^2+y^2)^8} + \frac{(x^2+y^2)^4 - (y)(4(x^2+y^2)^3(2y)}{(x^2+y^2)^8} \\
&= \frac{x^2+y^2-8x^2}{(x^2+y^2)^5} + \frac{x^2+y^2-8y^2}{(x^2+y^2)^5} \\
&= \frac{-6x^2-6y^2)}{(x^2+y^2)^5} \;=\; \frac{-6}{(x^2+y^2)^4}
\end{aligned}$$

which is negative for all $(x,y) \neq (0,0)$. Geometrically, it is because the arrows pointing into any point (again, other than the origin) are bigger than the arrows pointing away from it.

I had intended to show a field that had positive divergence at $(0,0)$ but smaller positive divergence away from it. A better example would have been

$$\mathbf{G}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{1+x^2+y^2}.$$

Here the calculation comes out as

$$(\nabla \cdot \mathbf{G})(x,y) = \frac{2}{(1+x^2+y^2)^2}$$

which is positive for all $(x,y) \in \mathbb{R}^2$, but greatest at $(0,0)$.

[6.2] #8: Let $\mathbf{F}(x,y) = 3xy\mathbf{i} + 2x^2\mathbf{j}$.

First, we evaluate $\oint_C \mathbf{F} \cdot d\mathbf{s}$ directly. We need to parametrize C. Let L, B, R be the left, bottom and right line segments, and let T be the semicircle on the top. Note that T has radius 1 and center at $(1,0)$, so it satisfies the equation $(x-1)^2 + y^2 = 1$. We can therefore parametrize the curves as

Curve	$\mathbf{x}(t)$	Range for t	$\mathbf{x}'(t)$
L	$(0,\ 2-t)$	$0 \leqslant t \leqslant 2$	$(0,-1)$
B	$(t,\ -2)$	$0 \leqslant t \leqslant 2$	$(1,0)$
R	$(2,\ t)$	$-2 \leqslant t \leqslant 0$	$(0,1)$
T	$(1+\cos t,\ \sin t)$	$0 \leqslant t \leqslant \pi$	$(-\sin t,\ \cos t)$

Therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_L \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt \quad + \quad \int_B \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt \quad + \quad \int_R \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt \quad + \quad \int_T \mathbf{F}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt$$

$$= \int_0^2 (0,0)\cdot(0,-1)\,dt \quad + \quad \int_0^2 (-6t, 2t^2)\cdot(1,0)\,dt \quad + \quad \int_{-2}^0 (6t,8)\cdot(0,1)\,dt$$

$$+ \int_0^\pi (3(1+\cos t)(\sin t)\mathbf{i} + 2(1+\cos t)^2\mathbf{j})\cdot(-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$$

$$= -\int_0^2 6t\,dt \quad + \quad \int_{-2}^0 8\,dt \quad + \quad \int_0^\pi (5\cos^3 t + 7\cos^2 t - \cos t - 3)\,dt$$

$$= -3t^2\Big|_0^2 \quad + \quad 8t\Big|_{-2}^0 \quad + \quad \int_0^\pi (7\cos^2 t - 3)\,dt \qquad \text{(by symmetry)}$$

$$= -12 + 16 + \left.\frac{t + 7\cos t\sin t}{2}\right|_0^\pi$$

$$= \boxed{4 + \frac{\pi}{2}.}$$

Meanwhile, let D be the area enclosed by the curve C. Green's Theorem says that

$$\oint_C \mathbf{F}\cdot d\mathbf{s} \;=\; \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA \;=\; \iint_D \left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(3xy)\right)dA \;=\; \iint_D (4x - 3x))\,dA$$

$$= \int_0^2 \int_{-2}^{\sqrt{1-(x-1)^2}} x\,dy\,dx$$

$$= \int_0^2 (2 + \sqrt{1-(x-1)^2})x\,dx$$

$$= \text{(calculation omitted)}$$

$$= \left.\frac{\arcsin(x-1)}{2} + x^2 + \frac{(x+1)(2x-3)\sqrt{2x-x^2}}{6}\right|_0^2$$

$$= 4 + \frac{\pi}{2}.$$

Note: I used a computer algebra system to do this last integral. Nothing this complicated will appear on Friday's test!

[6.2] #10: Call the ellipse C and call the region it encloses D. The work done by the field on the particle is

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA$$

$$= \iint_D \left(\frac{\partial}{\partial x}(x-4y) - \frac{\partial}{\partial y}(4y-3x)\right)dA$$

$$= \iint_D -3\,dA$$

$$= 3(\text{area of } D).$$

We have shown in class (and see the book, Example 3, p.430) that the area of an ellipse with horizontal and vertical radii a, b is πab. Therefore the area of D is 2π and the integral is -6π.