Example: The moment curve with position vector $\mathbf{x} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$.

Velocity $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$ **Speed** $\|\mathbf{v}\| = (1 + 4t^2 + 9t^4)^{1/2} = \dfrac{ds}{dt}$

Acceleration $\mathbf{a} = 0\mathbf{i} + 2\mathbf{j} + 6t\mathbf{k}$ **Arclength** $s = \displaystyle\int_0^t (1 + 4u^2 + 9u^4)^{1/2}\, du$

Jerk $\mathbf{a}' = 0\mathbf{i} + 0\mathbf{j} + 6\mathbf{k}$ (integral has no closed form)

Unit tangent vector, its time derivative and its magnitude:

$$\mathbf{T} = \frac{\mathbf{i} \ + \ 2t\mathbf{j} \ + \ 3t^2\mathbf{k}}{(1 + 4t^2 + 9t^4)^{1/2}}$$

$$\mathbf{T}' = \frac{-2t(2 + 9t^2)\mathbf{i} \ + \ 2(1 - 3t^2)(1 + 3t^2)\mathbf{j} \ + \ 6t(2t^2 + 1)\mathbf{k}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

$$\|\mathbf{T}'\| = \frac{2(1 + 9t^2 + 9t^4)^{1/2}}{1 + 4t^2 + 9t^4}$$

Curvature:

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt}\right\| \Big/ \left|\frac{ds}{dt}\right| = \frac{2(1 + 9t^2 + 9t^4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Alternate formula for curvature:

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\|6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}\|}{((1 + 4t^2 + 9t^4)^{1/2})^3} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}} = \frac{2(9t^4 + 9t^2 + 1)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Unit normal vector:

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{-t(2 + 9t^2)\mathbf{i} \ + \ (1 - 9t^4)\mathbf{j} \ + \ 3t(2t^2 + 1)\mathbf{k}}{(1 + 9t^2 + 9t^4)^{1/2}(1 + 4t^2 + 9t^4)^{1/2}}$$

Unit binormal vector:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{(1 + 9t^2 + 9t^4)^{1/2}(1 + 4t^2 + 9t^4)} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ -t(2 + 9t^2) & (1 - 9t^4) & 3t(2t^2 + 1) \end{vmatrix}$$

$$= \frac{3t^2\mathbf{i} \ - \ 3t\mathbf{j} \ + \ \mathbf{k}}{1 + 9t^2 + 9t^4}$$

Torsion:

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{3t(2 + 9t^2)\mathbf{i} \ + \ 3(9t^4 - 1)\mathbf{j} \ - \ 9t(2t^2 + 1)\mathbf{k}}{(1 + 9t^2 + 9t^4)^{3/2}(1 + 4t^2 + 9t^4)^{1/2}}$$

$$= -\underbrace{\left(\frac{3}{1 + 9t^2 + 9t^4}\right)}_{\tau} \underbrace{\left(\frac{-t(2 + 9t^2)\mathbf{i} \ - \ (9t^4 - 1)\mathbf{j} \ + \ 3t(2t^2 + 1)\mathbf{k}}{(1 + 9t^2 + 9t^4)^{1/2}(1 + 4t^2 + 9t^4)^{1/2}}\right)}_{\mathbf{N}}$$

Alternate formula for torsion:

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{\big(6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}\big) \cdot (6\mathbf{k})}{\|6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}\|^2} = \frac{12}{36t^4 + 36t^2 + 4} = \frac{3}{9t^4 + 9t^2 + 1}.$$

Example: Colley §3.2 #19

$$\mathbf{x} = \left(t,\ \frac{1}{3}(t+1)^{3/2},\ \frac{1}{3}(1-t)^{3/2}\right)$$

$$\mathbf{v} = \left(1,\ \frac{(1+t)^{1/2}}{2},\ -\frac{(1-t)^{1/2}}{2}\right) \qquad \|\mathbf{v}\| = \frac{ds}{dt} = \sqrt{1+\frac{1+t}{4}+\frac{1-t}{4}} = \sqrt{3/2}$$

$$\mathbf{a} = \left(0,\ \frac{(1+t)^{-1/2}}{4},\ \frac{(1-t)^{-1/2}}{4}\right) \qquad s = \int_0^t \sqrt{3/2}\,du = t\sqrt{3/2}$$

$$\mathbf{a}' = \left(0,\ \frac{-(1+t)^{-3/2}}{8},\ \frac{(1-t)^{-3/2}}{8}\right)$$

Unit tangent vector, its time derivative and its magnitude:

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}\left(1,\ (t+1)^{1/2},\ -(1-t)^{1/2}\right)$$

$$\mathbf{T}' = \frac{\sqrt{2}}{4\sqrt{3}}\left(0,\ (t+1)^{-1/2},\ (1-t)^{-1/2}\right)$$

$$\|\mathbf{T}'\| = \frac{\sqrt{2}}{4\sqrt{3}}\sqrt{\frac{1}{t+1}+\frac{1}{1-t}} = \frac{\sqrt{2}}{4\sqrt{3}}\sqrt{\frac{2}{1-t^2}} = \frac{1}{2\sqrt{3}\sqrt{1-t^2}}$$

Curvature:

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt}\right\| \Big/ \left|\frac{ds}{dt}\right| = \|\mathbf{T}'\|/\|\mathbf{v}\| = \frac{\sqrt{2}}{6\sqrt{1-t^2}}$$

Alternate formula for curvature:

$$\kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\left\|\left(\frac{1}{4(1-t^2)},\ \frac{-1}{4(1-t)^{1/2}},\ \frac{1}{4(t+1)^{1/2}}\right)\right\|}{3^{3/2}2^{-3/2}}$$

$$= \frac{2\sqrt{2}}{3\sqrt{3}}\sqrt{\frac{1}{16(1-t^2)^2}+\frac{1}{16(1-t)}+\frac{1}{16(t+1)}}$$

$$= \frac{\sqrt{2}}{6\sqrt{1-t^2}} \qquad \text{(after lots of algebra)}$$

Unit normal vector:

$$\mathbf{N} \;=\; \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \;=\; \frac{1}{\sqrt{2}}\Big(0,\ (1-t)^{1/2},\ (1+t)^{1/2}\Big)$$

Unit binormal vector:

$$\mathbf{B} \;=\; \mathbf{T}\times\mathbf{N} \;=\; \frac{1}{\sqrt{3}}\Big(1,\ -(1+t)^{1/2},\ (1-t)^{1/2}\Big)$$

Torsion:

$$\begin{aligned}\frac{d\mathbf{B}}{ds} \;&=\; \frac{d\mathbf{B}/dt}{ds/dt} = \frac{1}{\sqrt{3}}\left(0,\ -\frac{1}{2}(1+t)^{-1/2}, -\frac{1}{2}(1-t)^{-1/2}\right)\ \Big/\ \frac{\sqrt{3}}{\sqrt{2}} \\ &= -\underbrace{\frac{1}{3(1+t)^{1/2}(1-t)^{1/2}}}_{\tau}\underbrace{\left(\frac{1}{\sqrt{2}}\right)\Big(0,\ (1-t)^{1/2}, (1+t)^{1/2}\Big)}_{\mathbf{N}}\end{aligned}$$

$$\tau \;=\; \frac{1}{3(1+t)^{1/2}(1-t)^{1/2}} \;=\; \frac{1}{3(1-t^2)^{1/2}}$$

Alternate formula for torsion:

$$\tau \;=\; \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{\|\mathbf{v}\times\mathbf{a}\|^2} \;=\; \frac{\left(\dfrac{1}{4(1-t^2)},\ \dfrac{-1}{4(1-t)^{1/2}},\ \dfrac{1}{4(t+1)^{1/2}}\right)\cdot\left(0,\ \frac{-(1+t)^{-3/2}}{8},\ \frac{(1-t)^{-3/2}}{8}\right)}{\dfrac{1}{16(1-t^2)^2}+\dfrac{1}{16(1-t)}+\dfrac{1}{16(t+1)}}$$

$$=\; \frac{\left(\dfrac{1}{16(1-t)^{3/2}(1+t)^{3/2}}\right)}{\left(\dfrac{3}{16(1-t^2)}\right)} \;=\; \frac{1-t^2}{3(1-t^2)^{3/2}} \;=\; \frac{1}{3(1-t^2)^{1/2}}$$