Units for Curvature and Torsion

An excellent question came up in class on 10/11: **What are the units of curvature and torsion?**

The short answer is **inverse length**. Here are several reasons why this makes sense.

Let's measure length in meters (m) and time in seconds (sec). Then the units for curvature and torsion are both m^{-1}.

Explanation #1 (quick-and-dirty, and at least makes sense for curvature): As you probably know, the curvature of a circle of radius r is $1/r$. In other words, if you expand a circle by a factor of k, then its curvature shrinks by a factor of k. This is consistent with the units of curvature being inverse-length. You can also check that if you scale $\mathbf{x}$ by a factor of k, then the torsion also gets scaled by a factor of $1/k$.

Explanation #2 (intuitive, geometric): Since curvature and torsion are both supposed to be intrinsic to a curve, and independent of the speed with which you move along it, their units should not involve time.

On the other hand, as you zoom in on a curve (i.e., the bigger a single unit if length looks to the naked eye), the more it looks like a line (assuming that $\mathbf{x}$ is differentiable), and lines have zero curvature and torsion.

Explanation #3 (relies on nontrivial formulas, but at least precise):

- The formula for curvature is

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

which implies that

$$\begin{aligned} \text{units for } \kappa &= \frac{(\text{units for } \mathbf{v})(\text{units for } \mathbf{a})}{(\text{units for } \mathbf{v})^3} = \frac{(\mathrm{m/sec})(\mathrm{m/sec}^2)}{(\mathrm{m/sec})^3} \\ &= \mathrm{m}^{-1}. \end{aligned}$$

- Meanwhile, the formula for torsion is

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \dot{\mathbf{a}}}{\|\mathbf{v} \times \mathbf{a}\|^2}$$

(mentioned but not proved in class; it's problem #31 on p. 207). This implies that

$$\begin{aligned} \text{units for } \tau &= \frac{(\text{units for } \mathbf{v})(\text{units for } \mathbf{a})(\text{units for } \dot{\mathbf{a}})}{(\text{units for } \mathbf{v})^2(\text{units for } \mathbf{a})^2} = \frac{(\mathrm{m/sec})(\mathrm{m/sec}^2)(\mathrm{m/sec}^3)}{(\mathrm{m/sec})^2(\mathrm{m/sec})^2} \\ &= \mathrm{m}^{-1}. \end{aligned}$$

Explanation #4 (most detailed, but also most general): How do units work in general? Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vectors with units (u,u,u) and (v,v,v) respectively. What are the units of things like $\|\mathbf{x}\|$, $\mathbf{x} \cdot \mathbf{y}$, and $\mathbf{x} \times \mathbf{y}$?

Units of $\|\mathbf{x}\|$: u.

For example, velocity $\mathbf{v}$ is a vector whose components all have units m/sec. Its magnitude $\|\mathbf{v}\|$ is speed, which is a scalar quantity with units m/sec. This is also consistent with the formula $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Units of a unit vector: None — they are pure numbers.

A unit vector represents a direction and is independent of length. Intuitively, a direction is like an angle in having no units. Algebraically, the way that we usually construct a unit vector is by taking some vector $\mathbf{y}$ and normalizing it to $\mathbf{y}/\|\mathbf{y}\|$. Whatever the units of $\mathbf{y}$ are, they cancel out.

Units of $\mathbf{x} \cdot \mathbf{y}$: uv.

Algebraically, we multiply components of $\mathbf{x}$ by components of $\mathbf{y}$ to get $\mathbf{x} \cdot \mathbf{y}$. Also, the formula $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$ says that the units of $\mathbf{x} \cdot \mathbf{x}$ are (the units of $\mathbf{x}$) squared.

Units of $\mathbf{x} \times \mathbf{y}$: (uv,uv,uv).

Again, this works out algebraically. Geometrically, $\|\mathbf{x} \times \mathbf{y}\|$ is the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$, so the units should multiply.

Units of $\partial\alpha/\partial\beta$ = units of α / units of β.

This makes sense since $\partial\alpha/\partial\beta$ represents the rate of change if α with respect to β.

Using these tools, we can figure out the units for all the quantities that describe a parametrized curve:

Name	Symbol/Formula	Units
position	$\mathbf{x}$	m
velocity	$\mathbf{v} = \dot{\mathbf{x}}$	m/ sec
acceleration	$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{x}}$	m/ $\sec^2$
jerk	$\dot{\mathbf{a}} = \dddot{\mathbf{x}}$	m/ $\sec^3$
arclength	$s = \|v\|$	m
unit tangent vector	$\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$	unitless
unit normal vector	$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\}}$	unitless
unit binormal vector	$\mathbf{B} = \mathbf{T} \times \mathbf{N}$	unitless
curvature	$\kappa = \frac{d\mathbf{T}}{ds}$	m^{-1}
torsion	$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$	(units of $\mathbf{B}$) / (units of s) / (units of $\mathbf{N}$) $= \mathrm{m}^{-1}$