Math 141 Honors Problems #13
Due date: Tuesday, 11/24/09

HP21 [3 points] Find a formula for

$$\int \boldsymbol{e^{ax}\sin bx\, dx}$$

in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ (where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers).

Let $I = \int e^{ax}\sin(bx)\, dx$.

Step 1: Apply integration by parts to I, with

$$u = e^{ax}, \qquad du = ae^{ax}\, dx,$$
$$dv = \sin(bx)\, dx, \qquad v = -(\cos(bx))/b$$

to get

$$I \;=\; -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\underbrace{\int e^{ax}\cos(bx)\, dx}_{J}.$$

Step 2: Apply integration by parts to J, with

$$u = e^{ax}, \qquad du = ae^{ax}\, dx,$$
$$dv = \cos(bx)\, dx, \qquad v = (\sin(bx))/b$$

to get

$$\begin{aligned} I \;&=\; -\frac{e^{ax}\cos(bx)}{b} \;+\; \frac{a}{b}\left[\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)\, dx\right] \\ &=\; -\frac{e^{ax}\cos(bx)}{b} \;+\; \frac{ae^{ax}\sin(bx)}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin(bx)\, dx \\ &=\; \underbrace{\left(-\frac{be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{b^2}\right)}_{K} - \frac{a^2}{b^2}I. \end{aligned}$$

Now, solving the equation $I = K - (a^2/b^2)I$ gives

$$I \;=\; \frac{b^2K}{a^2+b^2} \;=\; -\frac{be^{ax}\cos(bx) + ae^{ax}\sin(bx)}{a^2+b^2}.$$

HP22 [3 points] Let $p(x)$ be a polynomial of degree n, say

$$\boldsymbol{p(x) = \sum_{k=0}^{n} a_k x^k}$$

where $a_0, a_1, \ldots, a_k, \ldots, a_n$ are real numbers. (There was a typo in the assignment, where the summand was given as $a_n x^n$ instead of the correct $a_k x^k$.) Find a formula for

$$\boldsymbol{\int e^x p(x)\, dx}$$

in terms of the a_k's.

Step 1: Find a formula just in terms of p, p', p'', etc., without worrying about the coefficients a_k. Integrating by parts repeatedly (always with u equal to the polynomial part of the integral and $dv = e^x\, dx$), we get

$$\begin{aligned}\int e^x p(x)\, dx &= e^x p(x) - \int e^x p'(x)\, dx \\ &= e^x p(x) - e^x p'(x) + \int e^x p''(x)\, dx \\ &= e^x p(x) - e^x p'(x) + e^x p''(x) - \int e^x p'''(x)\, dx \\ &= \cdots\end{aligned}$$

This process stops after $n+1$ iterations, because $p^{(n+1)}(x) = 0$. We conclude that

$$\int e^x p(x)\, dx \;=\; e^x \left[p(x) - p'(x) + p''(x) - \cdots + (-1)^n p^{(n)}(x)\right] \;=\; e^x \sum_{i=0}^{n} (-1)^i p^{(i)}(x). \qquad (*)$$

Step 2: Plug in the a_k's. Observe that

$$\begin{aligned}p(x) &= \sum_{k=0}^{n} a_k x^k, \\ p'(x) &= \sum_{k=0}^{n} k a_k x^{k-1} \quad = \quad \sum_{j=0}^{n-1} (j+1) a_{j+1} x^j, \\ p''(x) &= \sum_{k=0}^{n} k(k-1) a_k x^{k-2} \quad = \quad \sum_{j=0}^{n-2} (j+2)(j+1) a_{j+2} x^j, \\ &\cdots \\ p^{(i)}(x) &= \sum_{k=0}^{n} k(k-1)\cdots(k-i+1) a_k x^{k-i} \\ &= \sum_{j=0}^{n-i} \big((j+i)(j+i-1)\cdots(j+2)(j+1)\big) a_{j+i} x^j.\end{aligned}$$

Plugging this into equation $(*)$ above, we get

$$\int e^x p(x)\, dx = e^x \sum_{i=0}^{n} (-1)^i \left[\sum_{j=0}^{n-i} \big((j+i)(j+i-1)\cdots(j+2)(j+1)\big) a_{j+i} x^j\right].$$

A note: The expression in big parentheses can be expressed more conveniently using factorials:

$$(j+i)(j+i-1)\cdots(j+2)(j+1) \;=\; \frac{(j+i)!}{j!}.$$

HP23 [4 points] As discussed in class, there are no closed formulas for $\int(e^x/x)\ dx$ or for $\int(e^x/x^2)\ dx$. On the other hand, there *are* similar-looking functions which can be antidifferentiated.

(23a) For which constants a, b, c can the integral

$$\int \left(\frac{ae^x}{x} + \frac{be^x}{x^2} + \frac{ce^x}{x^3} \right) dx$$

be evaluated?

Observe that

$$\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{xe^x - e^x}{x^2} = \frac{e^x}{x} - \frac{e^x}{x^2},$$

$$\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{x^2e^x - 2xe^x}{x^4} = \frac{e^x}{x^2} - \frac{2e^x}{x^3}.$$

The first equation implies that $a = 1$, $b = -1$, $c = 0$ is a solution (i.e., the corresponding integral can be evaluated), and the second equation implies that $a = 0$, $b = 1$, $c = -2$ is a solution.

On the other hand, more generally, if two triples (a, b, c) and (A, B, C) are both solutions, then so are things like $(a + A, b + B, c + C)$, and $(2a, 2b, 2c)$, and $(4a - 7A, 4b - 7B, 4c - 7C)$, etc. (Here's a sneak preview of linear algebra: the set of all solutions is what is called a *vector space*.)

The most general rule is that (a, b, c) is a solution if and only if

$$\boxed{a + b + c/2 = 0.}$$

(You can verify that both $(1, -1, 0)$ and $(0, 1, -2)$ satisfy this condition.)

(23b) Can you say anything more generally about integrals of the form

$$\int \left(\sum_{k=1}^{n} \frac{a_k e^x}{x^k} \right) dx?$$

The pattern in (23a) is the tip of the following iceberg: this integral can be evaluated if and only if

$$\sum_{k=1}^{n} \frac{a_k}{k!} = 0.$$

For example, the integral

$$\int e^x \left(\frac{3}{x} - \frac{2}{x^2} + \frac{1}{2x^3} + \frac{7}{x^4} \right) dx$$

cannot be evaluated because

$$\frac{3}{0!} - \frac{2}{1!} + \frac{1/2}{2!} + \frac{7}{3!} = 3 - 2 + \frac{1}{4} + \frac{7}{6} = \frac{29}{12} \neq 0,$$

On the other hand, $29/12 = 58/24 = 58/4!$, so the integral

$$\int e^x \left(\frac{3}{x} - 2\frac{x^2}{+}\frac{1}{2x^3} + \frac{7}{x^4} - \frac{58}{x^4} \right) dx$$

can be evaluated!