Math 141 Homework #9
Due Tuesday, 10/23/07 Extra Problems

Problem #1 Consider the function

$$b(x) = x^2 + \frac{\ln|x-2|}{1000}.$$

(#1a) Without using a calculator, sketch the graph of $b(x)$ for $-10 \leq x \leq 10$.

(#1b) Enter $b(x)$ into your calculator and have it draw the graph. The result will probably not look like the graph you drew in part 1 (at least if you are using a TI-83+ or something similar). Who's right, you or the calculator?

(#1c) Can you resolve this problem by changing the viewing window?

Problem #2 Read this article from the Lawrence Journal-World and write a sentence or two clearly explaining the mathematical basis for Sen. Haley's proposal.

Problem #3 (Bonus problem; challenging!) The Extreme Value Theorem states that if a function $f(x)$ is continuous on a closed interval I, then f achieves a global maximum and a global minimum on I. In class on Tuesday 10/16, we discussed the possibility that f has infinitely many critical numbers in I. Let's call a function *wild* if it exhibits this behavior, and *tame* otherwise. As we saw in class, an example of a wild function is

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1 \\ 0 \text{ if } x = 1 \end{cases}$$

on the interval $I - [0,1]$.

We know that if f is tame, then we can find the global minimum and maximum of f on I by listing all the (finitely many) critical numbers and endpoints, evaluating f at each of them, and comparing the values. We'd like to prove that this method still works even for wild functions.

(#3a) First, explain why the range of f must be an interval. That is, rule out the possibility that the range is something like

$$[-4,0) \cup (1,3]$$

or

$$[-4,0) \cup (0,3)$$

or

$$\{1,2,3,5,8,13,21\}.$$

The next step is to figure out whether the interval is open, closed or half-and-half, and whether it is finite or infinite.

(#3b) Second, prove the following Lemma. If $\{x_1, x_2, x_3, \dots\}$ is an *infinite* set of numbers in I, then there is some number $a \in I$ that is an "accumulation point" of the x_i's — that is, with the property that any open[1] interval containing a also contains at least one of the x_i's.

This Lemma is a key tool for the rest of the problem. If you don't see how to prove the Lemma, that's okay; you can still do the rest of the problem by assuming that the Lemma is true.

[1]Or possibly half-open, if a is an endpoint of I, but you can ignore that case if you want to.

(#3c) Now prove that f is *bounded* on I; that is, there are numbers A and B (for "above" and "below") such that

$$B \leq f(x) \leq A$$

for every $x \in I$. (Hint: Think about what would have to happen if f is *not* bounded, and use the Lemma.)

Another way of saying this result is that the range of f is a subset of the interval $[B, A]$, therefore a finite interval. We might as well assume that the interval is one of the following:

$$[B, A], \qquad (B, A], \qquad [B, A), \qquad (B, A).$$

(#3d) Rule out the possibility that the range is an open or half-open interval. (Hint: The numbers $(A + B)/2$, $(2A + B)/3$, $(3A + B)/4$, $\ldots$ all lie in the range; use this together with the Lemma.)