Complete Graphs

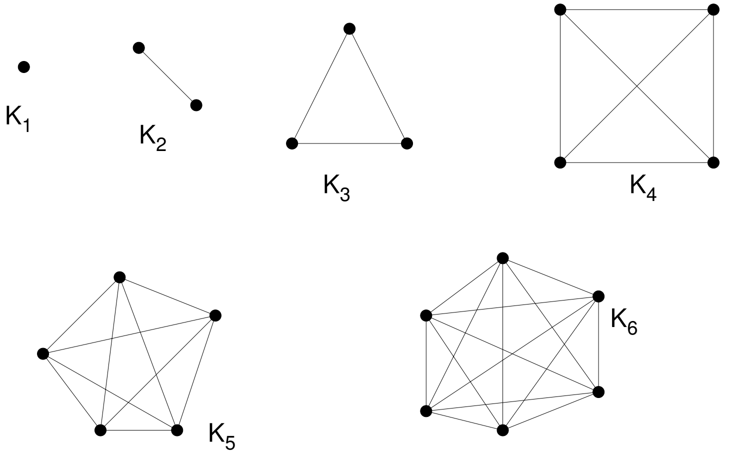
Let N be a positive integer.

Definition: A **complete graph** is a graph with N vertices and an edge between every two vertices.

- There are no loops.
- Every two vertices share exactly one edge.

We use the symbol K_N for a complete graph with N vertices.

Complete Graphs



Complete Graphs

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Complete Graphs

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- This formula **also** counts the **number of pairwise comparisons** between N candidates (recall §1.5).
- The Method of Pairwise Comparisons can be modeled by a complete graph.
  - Vertices represent candidates
  - Edges represent pairwise comparisons.
  - Each candidate is compared to each other candidate.
  - No candidate is compared to him/herself.

Hamilton Circuits in K_N

How many different Hamilton circuits does K_N have?

- Let's assume $N = 3$.

Hamilton Circuits in K_N

How many different Hamilton circuits does K_N have? ★

- Let's assume $N = 3$.
- We can represent a Hamilton circuit by listing all vertices of the graph in order.
- The first and last vertices in the list must be the same. All other vertices appear exactly once.
- We'll call a list like this an "itinerary".

Hamilton Circuits in K_N

How many different Hamilton circuits does K_N have?

Some possible itineraries:

$A, C, D, B, A \qquad Y, X, W, U, V, Z, Y \qquad Q, W, E, R, T, Y, Q$

- The first/last vertex is called the “reference vertex”.

Hamilton Circuits in K_N

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Some possible itineraries:

$A, C, D, B, A \qquad Y, X, W, U, V, Z, Y \qquad Q, W, E, R, T, Y, Q$

- The first/last vertex is called the "reference vertex".
- **Changing the reference vertex does not change the Hamilton circuit**, because the same edges are traveled in the same directions.
- That is, different itineraries can correspond to the same Hamilton circuit

Hamilton Circuits in K_N

Changing the reference vertex does not change the Hamilton circuit.

For example, these itineraries all represent the same Hamilton circuit in K_4:

A, C, D, B, A	(reference vertex: A)
B, A, C, D, B	(reference vertex: B)
D, B, A, C, D	(reference vertex: C)
C, D, B, A, C	(reference vertex: D)

Hamilton Circuits in K_N

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For example, these itineraries all represent the same Hamilton circuit in K_4:

A, C, D, B, A	(reference vertex: A)
B, A, C, D, B	(reference vertex: B)
D, B, A, C, D	(reference vertex: C)
C, D, B, A, C	(reference vertex: D)

Every Hamilton circuit in K_N **can be described by exactly N different itineraries** (since there are N possible reference vertices).

Hamilton Circuits in K_N

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Hamilton Circuits in K_N

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- N possibilities for the reference vertex

Hamilton Circuits in K_N

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Hamilton Circuits in K_N

So, how many possible itineraries are there?

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Hamilton Circuits in K_N

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Hamilton Circuits in K_N

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- 2 possibilities for the $(N-1)$st vertex

Hamilton Circuits in K_N

So, how many possible itineraries are there?

- N possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- $\dots$
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the Nth vertex

Hamilton Circuits in K_N

So, how many possible itineraries are there?

- N possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
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- ...
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the Nth vertex
- and then the reference vertex again.

Hamilton Circuits in K_N

So, how many possible itineraries are there?

- ~~N possibilities for the reference vertex~~
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- ...
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the Nth vertex
- and then the reference vertex again.

If we are counting Hamilton circuits, then we don't care about the reference vertex.

Hamilton Circuits in K_N

Conclusion: The number of Hamilton circuits in $\mathbf{K_N}$ is

$$\mathbf{(N-1) \times (N-2) \times \cdots \times 3 \times 2 \times 1 \quad = \quad \boxed{(N-1)!}}$$

Each one can be described by N different itineraries.

(So the number of <u>itineraries</u> is actually $N!$.)

Hamilton Circuits in K_N

For every $N \geq 3$,

The number of Hamilton circuits in $\mathbf{K_N}$ is $\mathbf{(N-1)!}$.

In comparison, for every $N \geq 1$,

The number of edges in $\mathbf{K_N}$ is $\mathbf{\dfrac{N(N-1)}{2}}$.

Hamilton Circuits in K_N

Vertices **N**	**Edges** $\mathbf{N(N-1)/2}$	**Hamilton circuits** $\mathbf{(N-1)!}$
1	0	
2	1	
3	3	2
4	6	6
5	10	24
6	15	120
7	21	620
...	...	...
16	120	1307674368000

Hamilton Circuits in K_3

Itineraries in K_3:

A,B,C,A	A,C,B,A
B,C,A,B	B,A,C,B
C,A,B,C	C,B,A,C

Hamilton Circuits in K_3

Itineraries in $\mathbf{K_3}$:

A,B,C,A	A,C,B,A
B,C,A,B	B,A,C,B
C,A,B,C	C,B,A,C

- Each column of the table gives 3 itineraries for the same Hamilton circuit (with different reference vertices).
- The number of Hamilton circuits is $(3-1)! = 2! = 2$.

Hamilton Circuits in K_4

Itineraries in $\mathbf{K_4}$:

ABCDA	ABDCA	ACBDA	ACDBA	ADBCA	ADCBA
BCDAB	BDCAB	BDACB	BACDB	BCADB	BADCB
CDABC	CABDC	CBDAC	CDBAC	CADBC	CBADC
DABCD	DCABD	DACBD	DBACD	DBCAD	DCBAD

- Each column lists 4 itineraries for the same Hamilton circuit.
- The number of Hamilton circuits is $(4-1)! = 3! = 6$.

Hamilton Circuits in K_4

Where have you seen this table before?

Hamilton Circuits in K_4

Where have you seen this table before?

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BCDA	BDCA	BDAC	BACD	BCAD	BADC
CDAB	CABD	CBDA	CDBA	CADB	CBAD
DABC	DCAB	DACB	DBAC	DBCA	DCBA

Hamilton Circuits in K_4

Where have you seen this table before?

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BCDA	BDCA	BDAC	BACD	BCAD	BADC
CDAB	CABD	CBDA	CDBA	CADB	CBAD
DABC	DCAB	DACB	DBAC	DBCA	DCBA

An itinerary (without the last vertex repeated) is the same thing as the list of sequential coalitions in a weighted voting system!

That's why there are $N!$ itineraries on N vertices.

Complete Graphs

By the way, for which values of N does the complete graph K_N have an Euler circuit?

Complete Graphs

By the way, for which values of N does the complete graph K_N have an Euler circuit?

Answer: When N is **odd**. (Every vertex in K_N has degree $N-1$, so we need $N-1$ to be even.)