The Seven Bridges of Königsberg

- In 1735, the city of Königsberg (present-day Kaliningrad) was divided into four districts by the Pregel River.[1]
- The four districts were connected by seven bridges.

[1] Source for Königsberg maps: MacTutor History of Mathematics archive, `www-history.mcs.st-and.ac.uk`

The Seven Bridges of Königsberg

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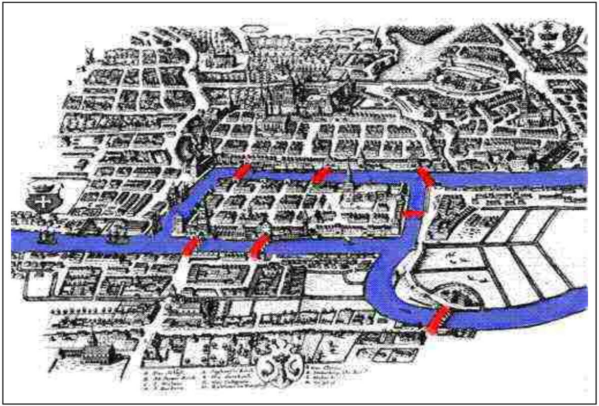
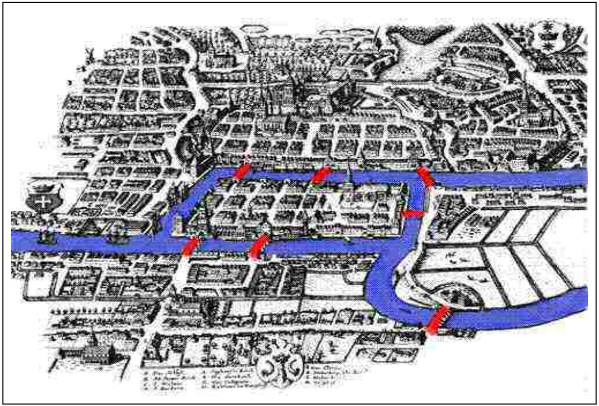


The Seven Bridges of Königsberg

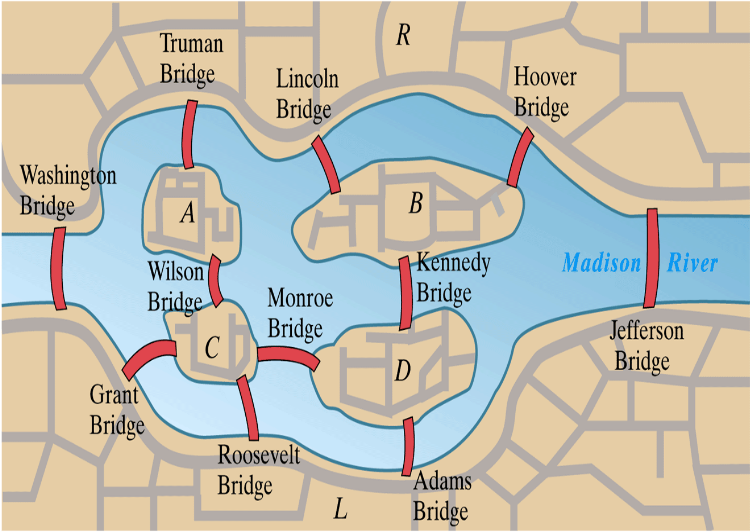


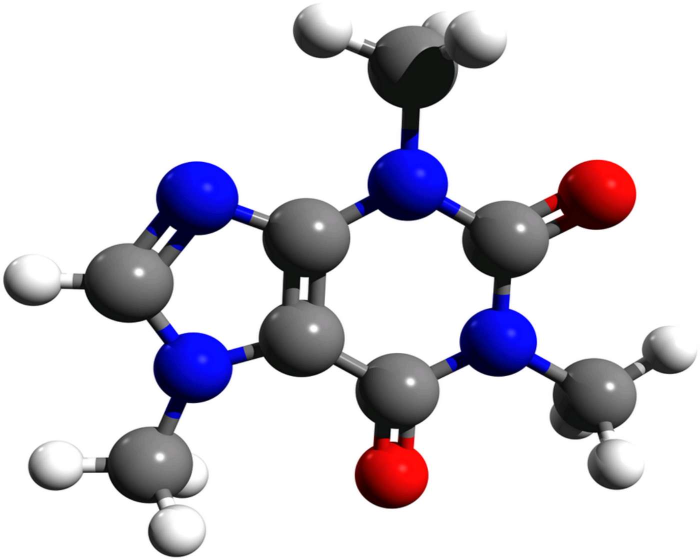
Is it possible to design a walking tour of Königsberg in which you cross each of the seven bridges **exactly once?**

The Mathematics of Networks

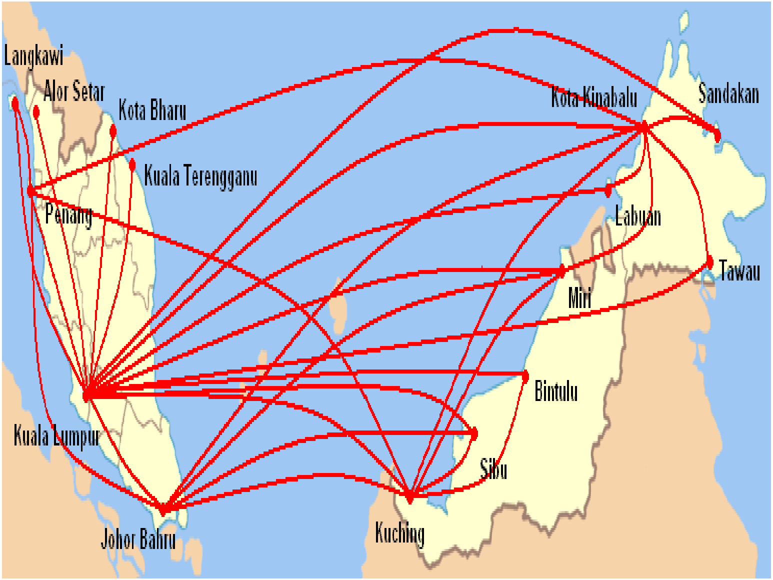
The mathematical models we need to solve the Königsberg problem is a **graph**.

- designing travel routes (Chapters 5, 6)
- connecting networks efficiently (Chapter 7)
- scheduling tasks (Chapter 8)
- coloring regions of maps (Mini-Excursion 2)

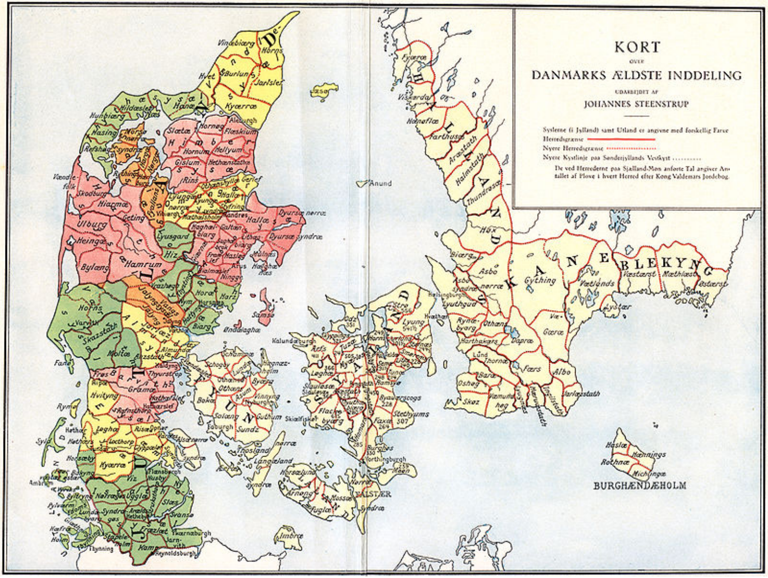

R
Truman Bridge
Lincoln Bridge
Hoover Bridge
Washington Bridge
A
B
Wilson Bridge
Kennedy Bridge
Madison River
Monroe Bridge
C
D
Jefferson Bridge
Grant Bridge
Roosevelt Bridge
L
Adams Bridge



Source: http://commons.wikimedia.org/wiki/File:Caffeine_3d_structure.png

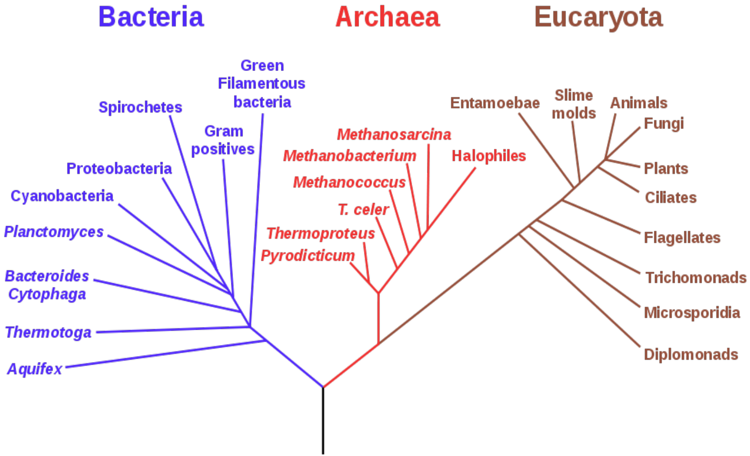


Source: upload.wikimedia.org/wikipedia/commons/2/20/AA_route_map.PNG

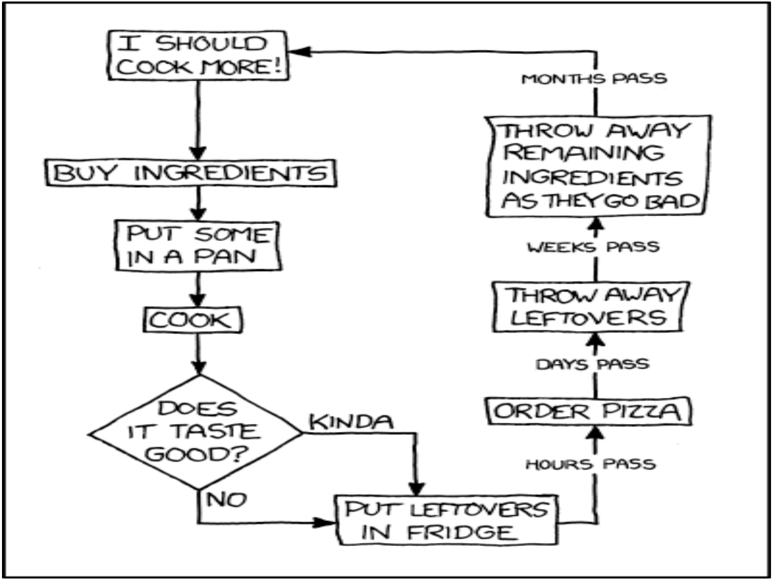


Source: `http://commons.wikimedia.org/wiki/Atlas_of_Denmark`

Phylogenetic Tree of Life



Source: `http://commons.wikimedia.org/wiki/File:Phylogenetic_tree.svg`



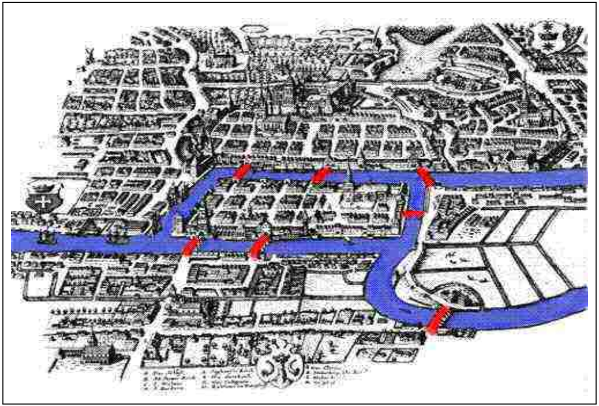
Source: `http://xkcd.com/854/`

Graphs

Things that can be modeled with graphs include

- maps
- molecules
- flow charts
- family trees
- Internet (web pages connected by links)
- Facebook/Google+ (people connected by friendship)
- Six Degrees of Kevin Bacon
- ...

The Königsberg Bridge Problem



Euler and the Königsberg Bridge Problem

The great Swiss mathematician Leonhard Euler (1707–1783) became interested in the Königsberg problem around 1735 and published a solution ("*Solutio problematis ad geometriam situs pertinentis*") in 1741.

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Euler's intuition: **The physical map doesn't matter.** What matters mathematically is just **the list of which regions are connected by bridges**.

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Euler's solution opened up an entire new branch of mathematics, now known as **graph theory**.

The Bridges of Königsberg

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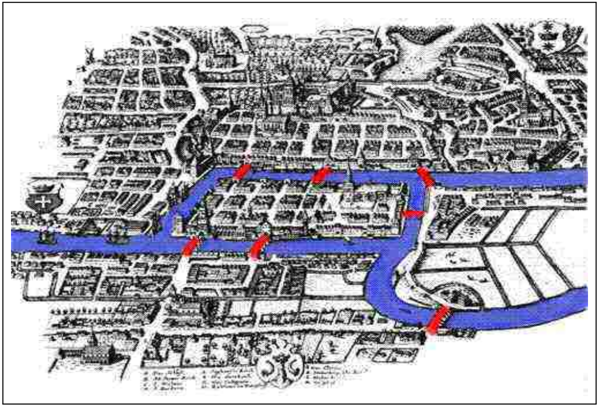
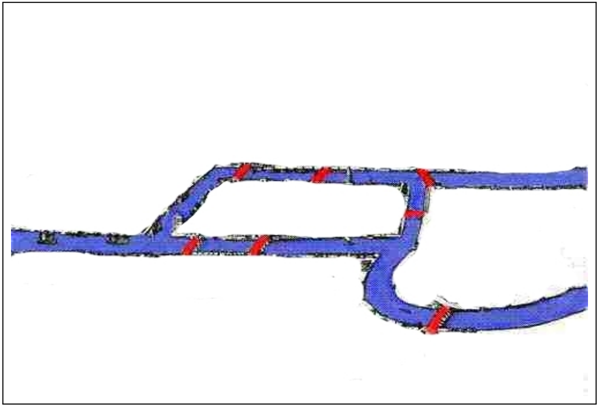


The Bridges of Königsberg



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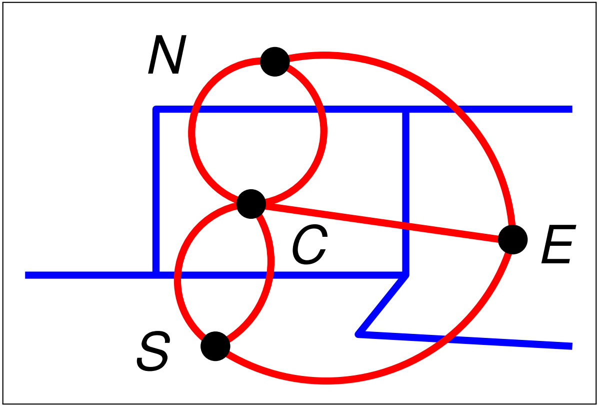
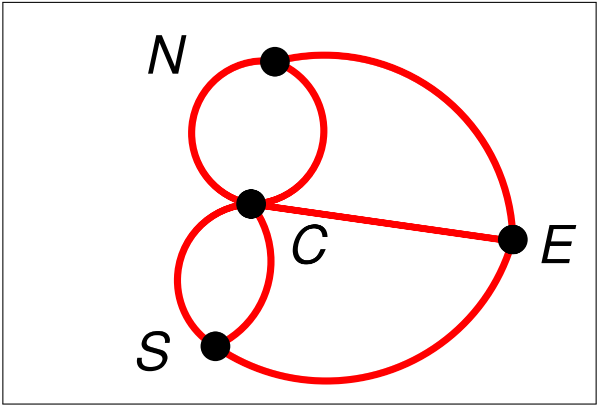


The Bridges of Königsberg



Euler and the Königsberg Bridge Problem

> *"...this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."*

(Leonhard Euler, letter of April 1736)

Euler and the Königsberg Bridge Problem

> *"This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it."*

(Leonhard Euler, letter of March 1736)

Routing Problems

The Bridges of Königsberg is an example of a **routing problem**.

Other examples:

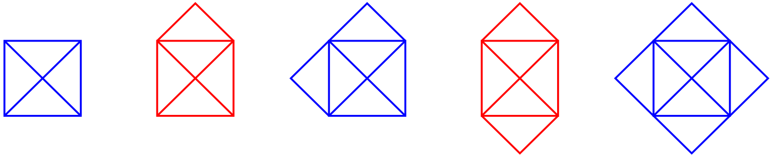
- Walking tour: must cross every bridge once
- Garbage collector: must visit every house once
- Airline traveler: get from Medicine Hat to Nairobi as cheaply as possible

Existence Question: Is an actual route possible?

Optimization Question: Which of all possible routes is the best? (I.e., the shortest, cheapest, most efficient, etc.)

Unicursal Tracings

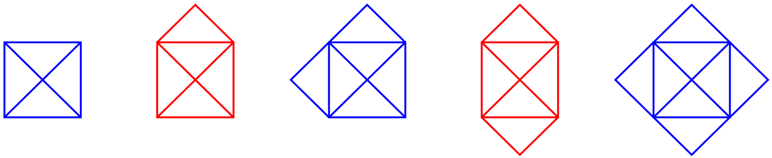
Which of these figures can you draw without ever lifting your pen from the page, or retracing a previous line?



For which ones can you finish with your pen at the same point it started?

Unicursal Tracings

Which of these figures can you draw without ever lifting your pen from the page, or retracing a previous line?



For which ones can you finish with your pen at the same point it started?

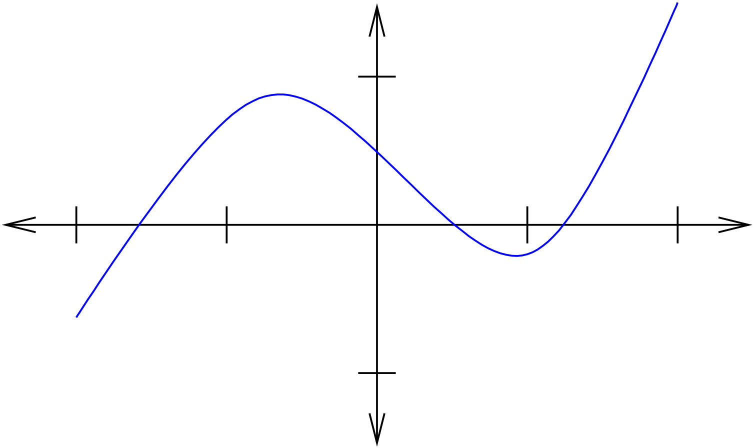
Mathematically, finding a unicursal tracing is equivalent to solving the Königsberg Bridge Problem!

Graphs

Definition: A **graph** consists of one or more **vertices**, attached by **edges**.

Frequently, we draw the vertices as points and the edges as line segments or curves.

This Is Not A Graph In Math 105



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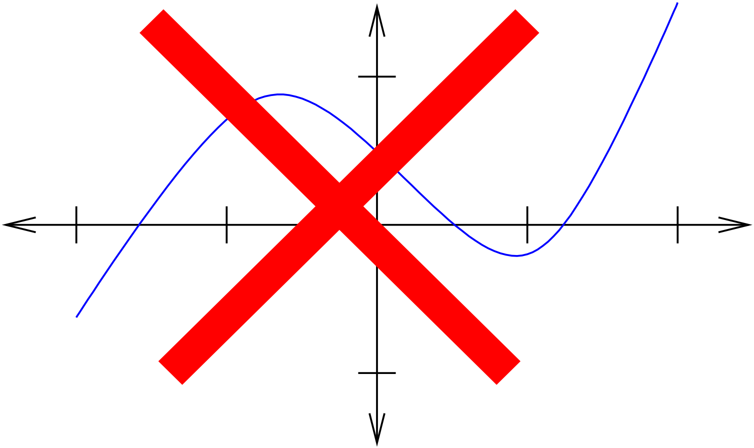


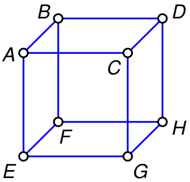
Graphs

Definition: A **graph** consists of one or more **vertices**, attached by **edges**.

Frequently, we draw the vertices as points and the edges as line segments or curves

However, it does not matter where the vertices are located on the page or what the edges are shaped like.

Different Representations of Graphs



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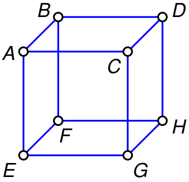

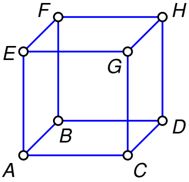

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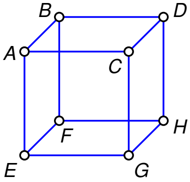

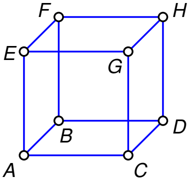

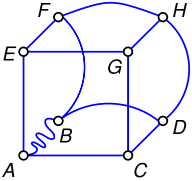

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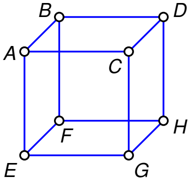

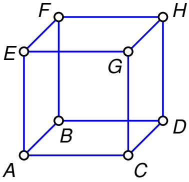

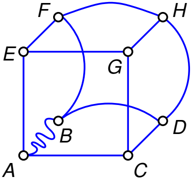

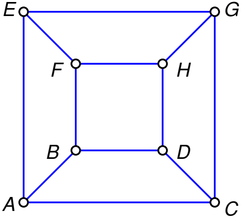

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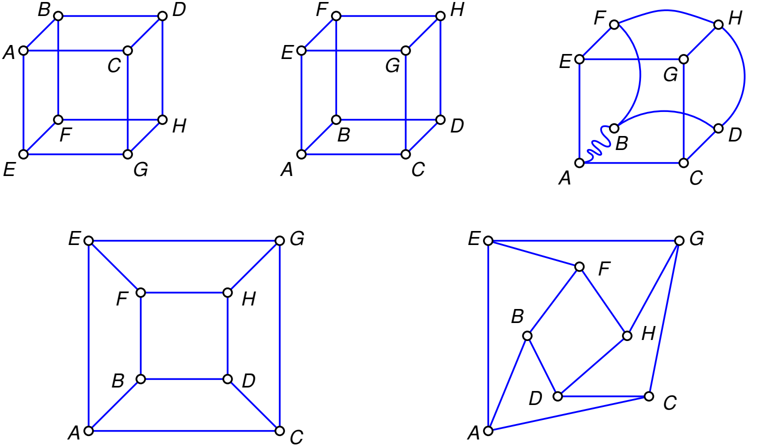

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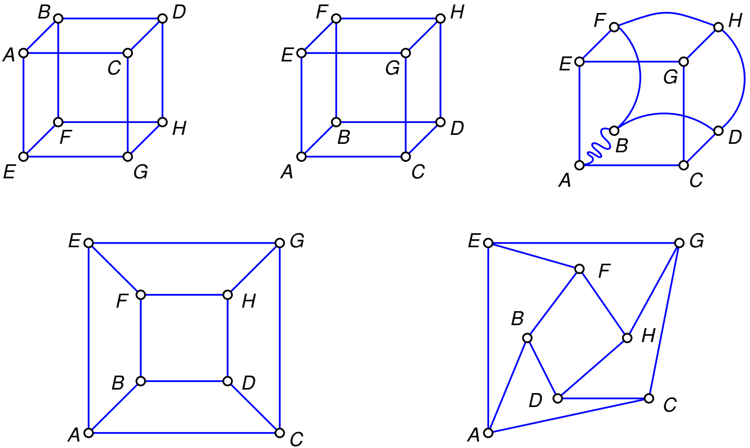
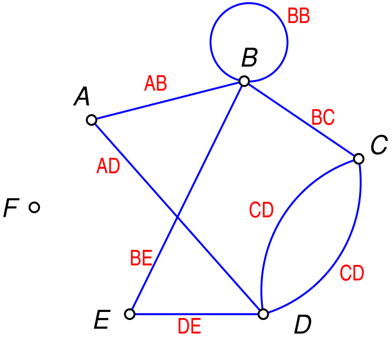


These five figures all represent the same graph!

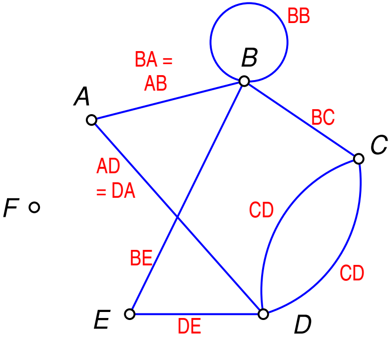
Graph Terminology



Vertex set:
$\mathcal{V} = \{A, B, C, D, E, F\}$

Edge set:
$\mathcal{E} = \{AB, BB, BC, CD, CD, DE, BE, AD\}$

Graph Terminology

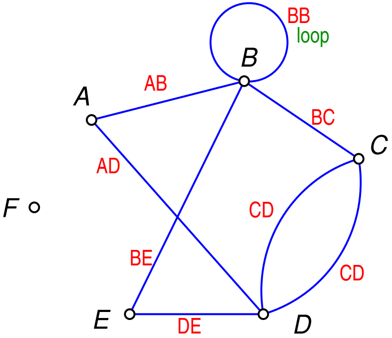


Vertex set:
$\mathcal{V} = \{A, B, C, D, E, F\}$

Edge set:
$\mathcal{E} = \{AB, BB, BC, CD, CD, DE, BE, AD\}$

- The order that we write the two vertices in an edge doesn't matter: *DE* and *ED* mean the same thing.

Graph Terminology

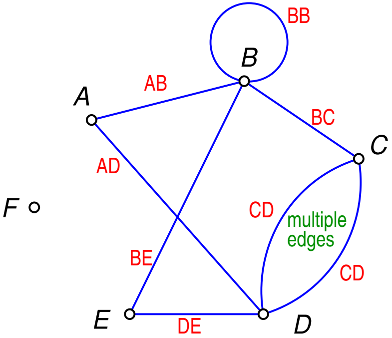


Vertex set:
$\mathcal{V} = \{A, B, C, D, E, F\}$

Edge set:
$\mathcal{E} = \{AB, BB, BC, CD, CD, DE, BE, AD\}$

- An edge can attach a vertex to itself (like BB); this is called a **loop**.

Graph Terminology



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Edge set:
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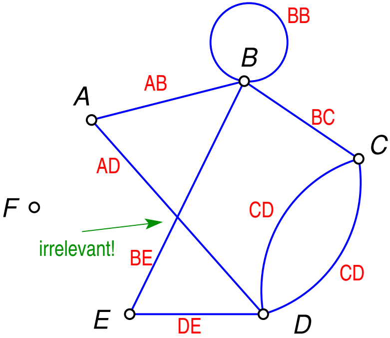
- There can be **multiple edges** between the same end-points (like CD, which is a double edge).

Graph Terminology



Vertex set:
$\mathcal{V} = \{A, B, C, D, E, F\}$

Edge set:
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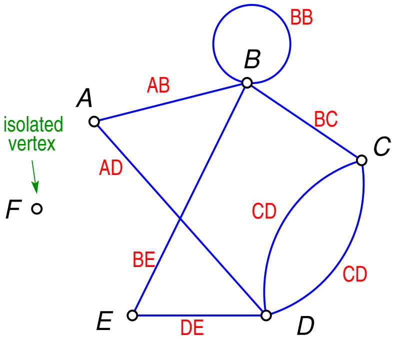
- It doesn't matter if edges cross each other; crossing points do **not** count as vertices.

Graph Terminology



Vertex set:
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Edge set:
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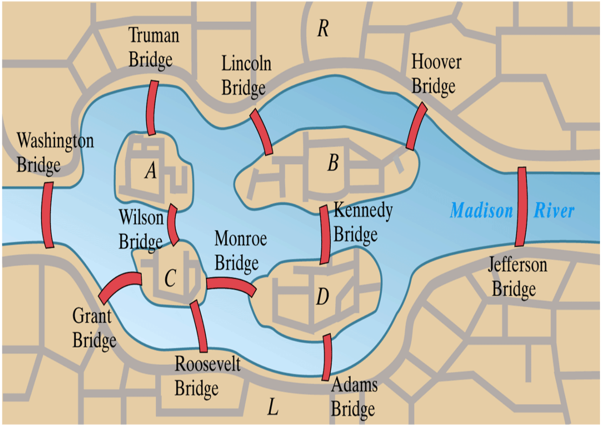
- Not every vertex has to have an edge attached to it. A vertex with no edges is called an **isolated vertex**.

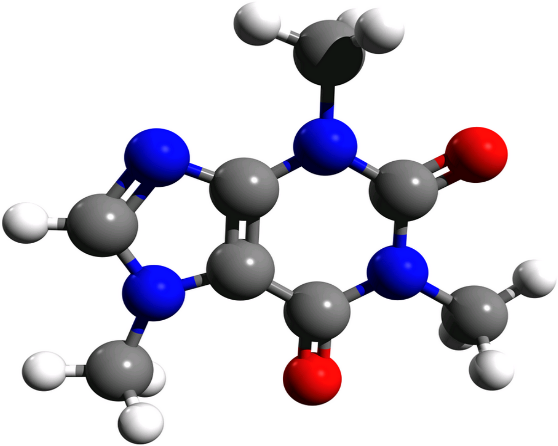
Graphs as Models

Graphs can be used as models for zillions of different structures arising in the real world.

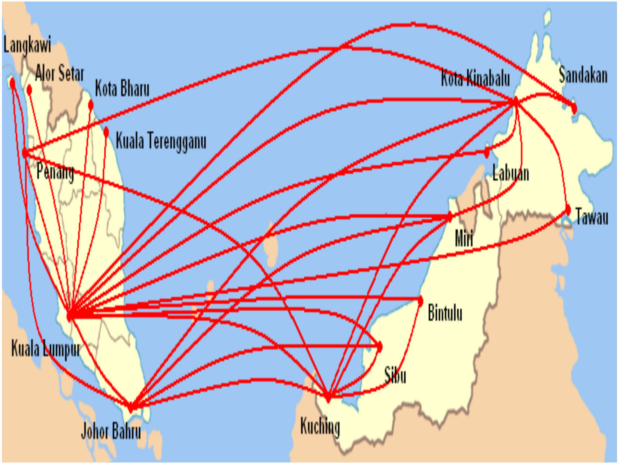
- **Facebook:** vertices = people, edges = friendships
- **Internet:** vertices = web pages, edges = links



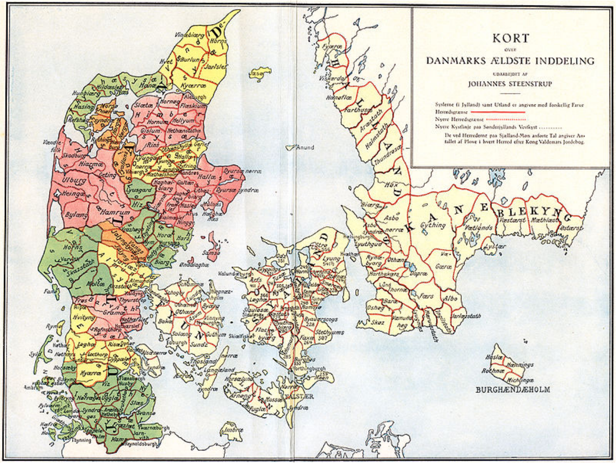
Vertices = regions of Königsberg; edges = bridges



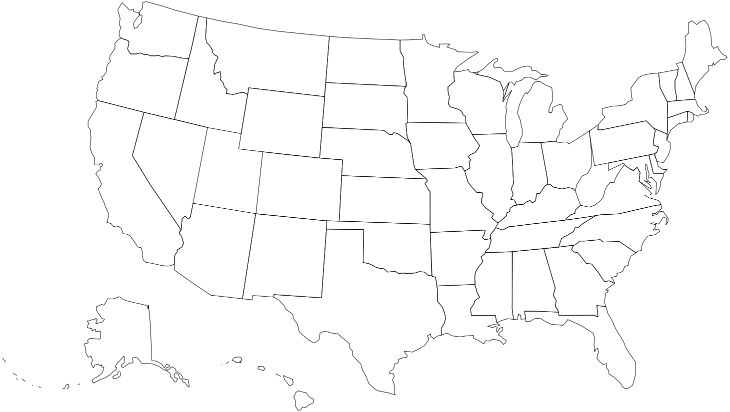
Vertices = atoms; edges = bonds

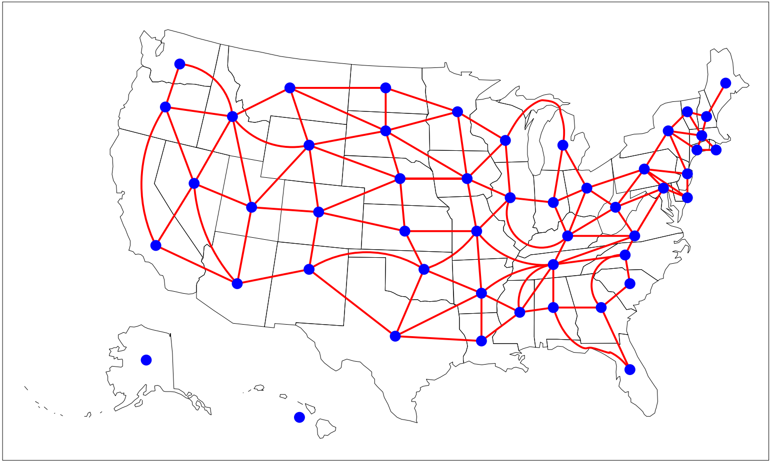


Vertices = cities; edges = airplane routes

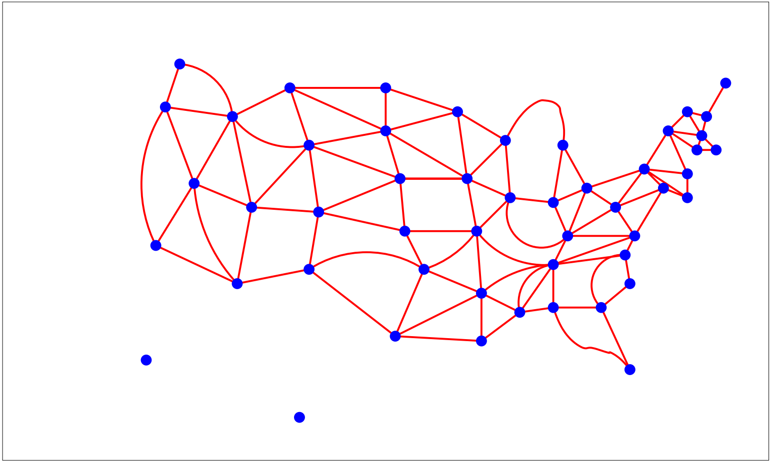


Vertices = regions of Denmark; edges = common borders





Vertices = states; edges = common borders



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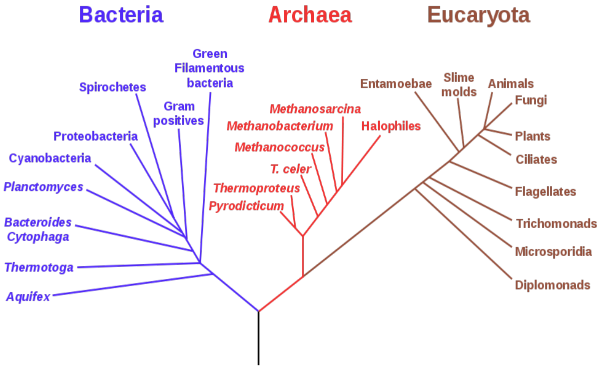
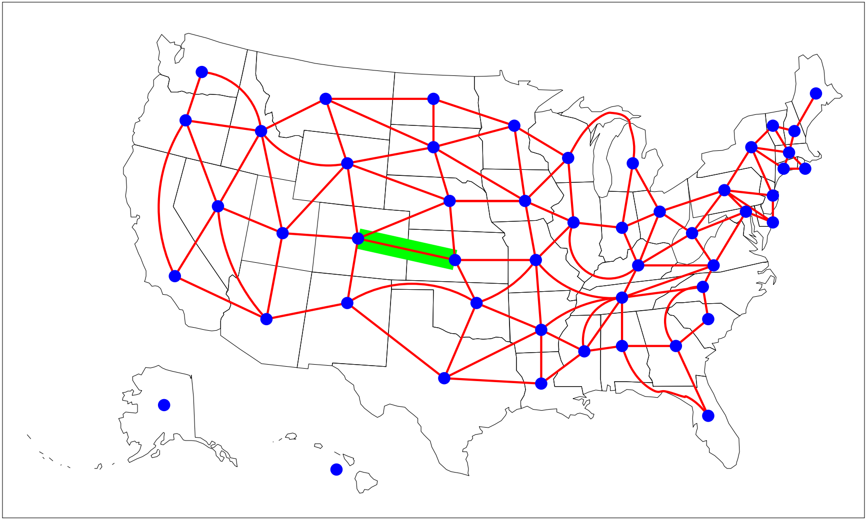
Phylogenetic Tree of Life



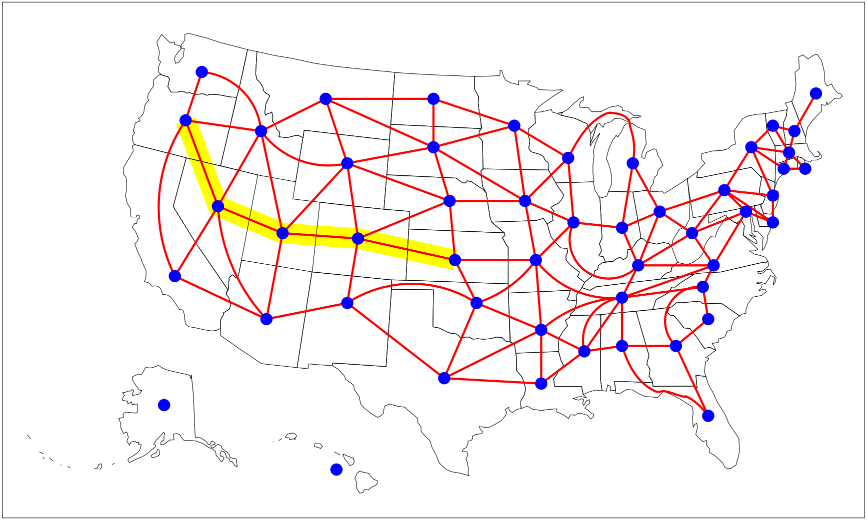
Vertices = groups of species; edges = biological kinship

Graph Terminology: Adjacency and Connectedness

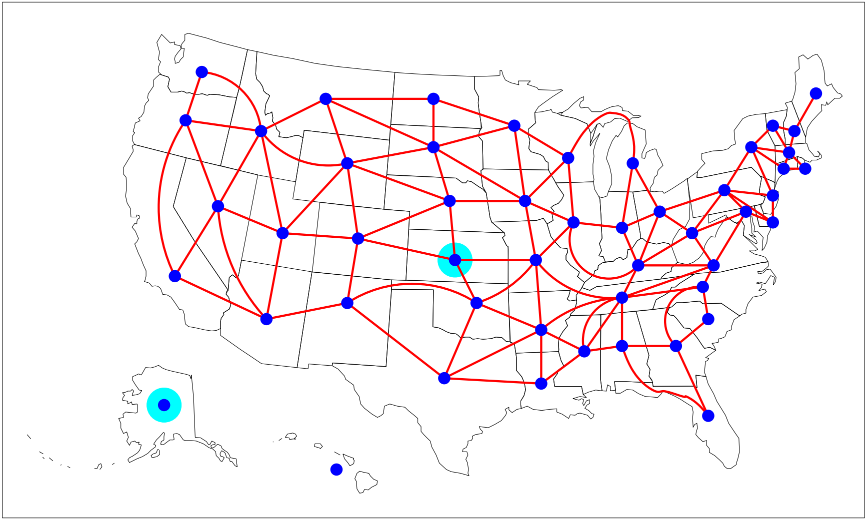
- Two vertices are called **adjacent** if they are attached directly by at least one edge.
- Two vertices are called **connected** if they are connected by a sequence of edges.
- Adjacent vertices are always connected, but connected vertices are not necessarily adjacent.



Kansas and Colorado are adjacent



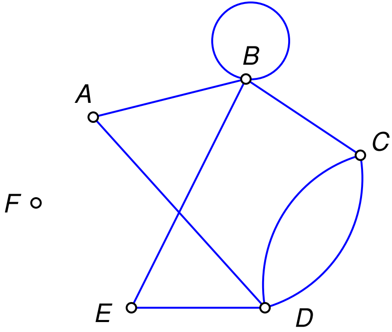
Kansas and Oregon are connected, but not adjacent



Kansas and Alaska are neither adjacent nor connected

Graph Terminology: Degree

The **degree** of a vertex v is the number of edges attached to v. (A loop counts as two edges.)



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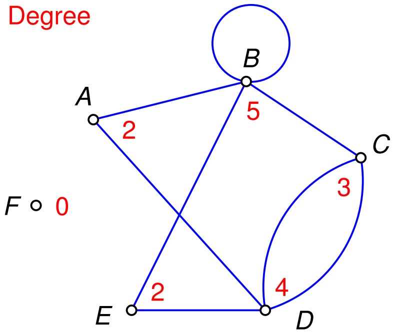

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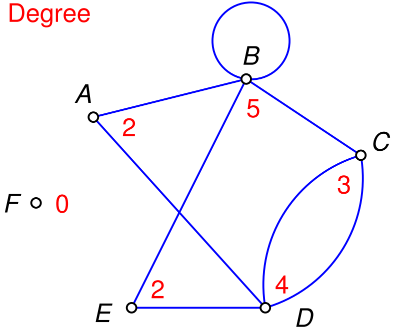
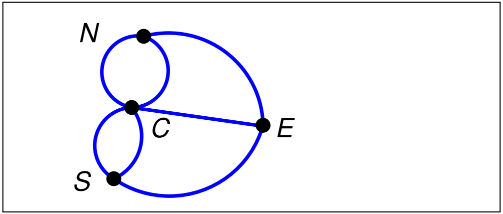


An **odd vertex** is a vertex whose degree is odd.
An **even vertex** is a vertex whose degree is even.

Graph Terminology: Degree

Example: Königsberg.
Vertices = regions of city
Degree of a vertex = number of bridges that go to that region



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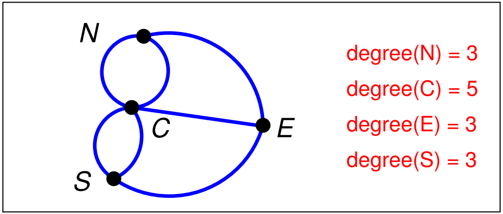
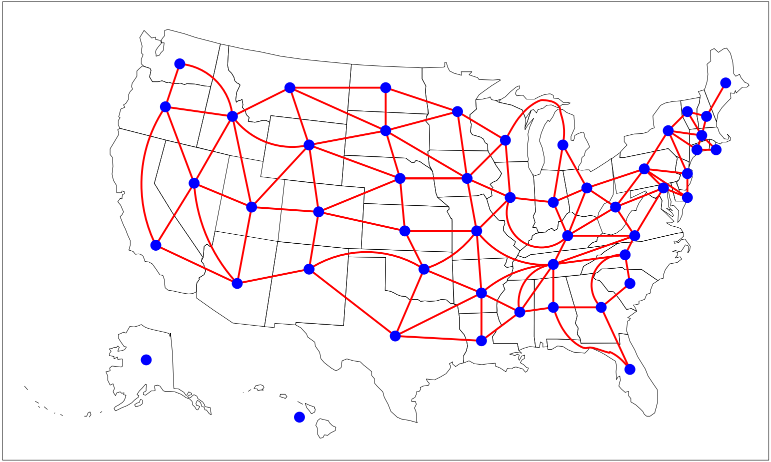


Graph Terminology: Degree

Example: Map of USA.

Vertices = states; degree = number of bordering states



Graph Terminology: Degree

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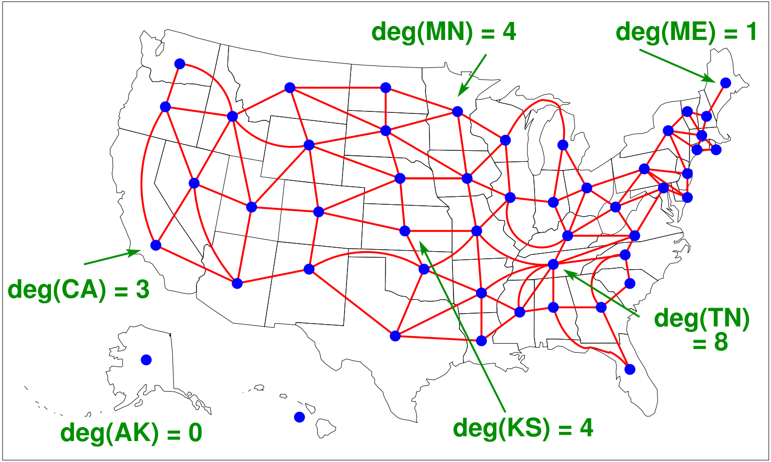
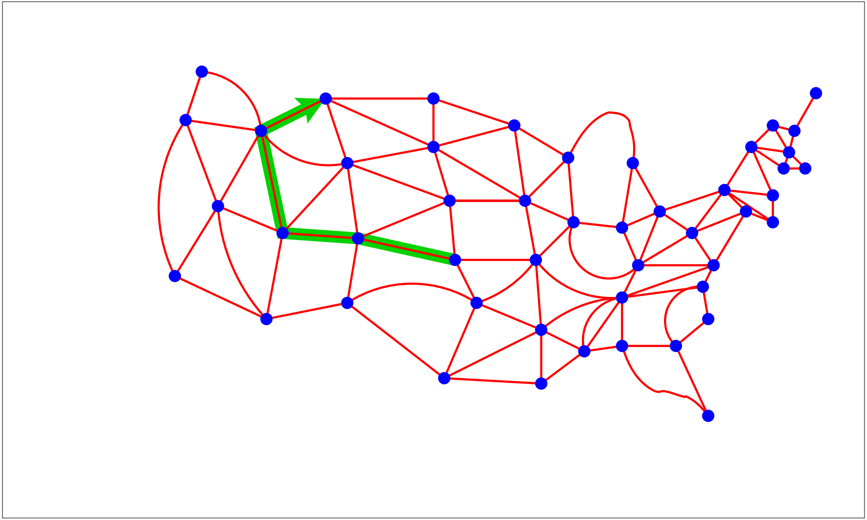
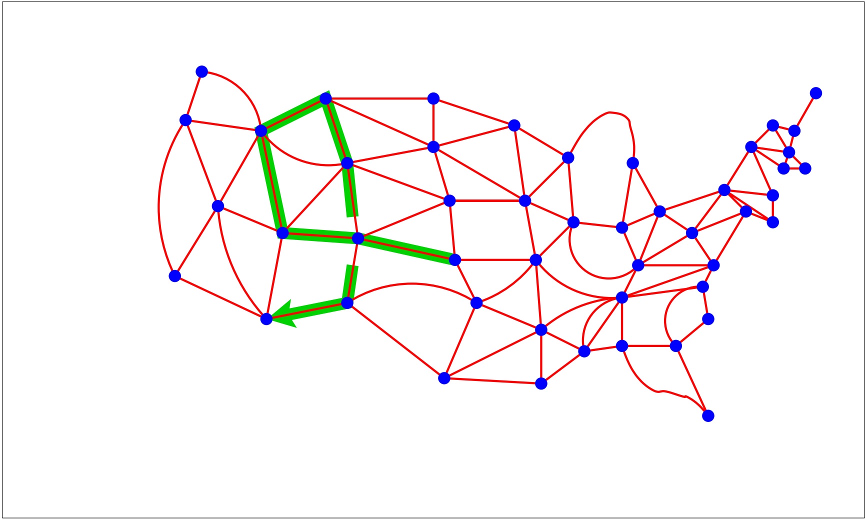


Paths and Circuits

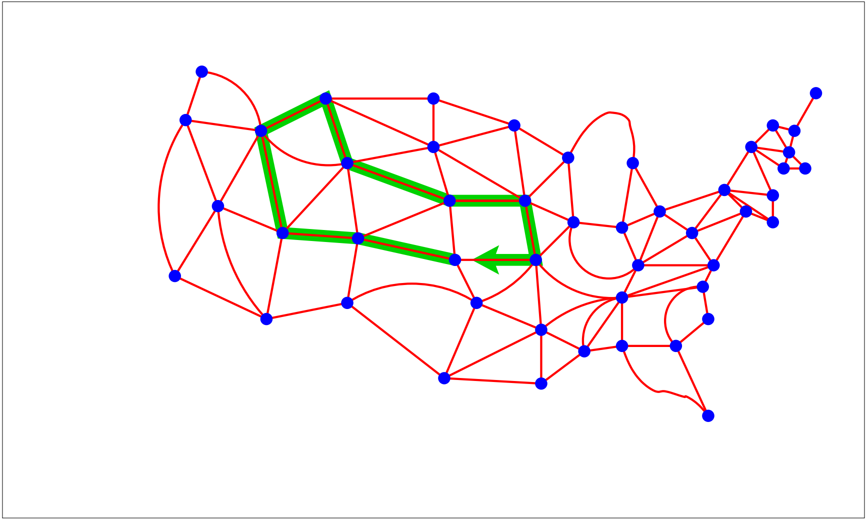
- In many graph models (maps, road networks, Kevin Bacon, . . .), we want to travel from one vertex to another by walking along the edges ("taking a trip").
- **Rule:** A trip cannot use the same **edge** more than once, but it may pass through the same **vertex** more than once.
- A trip is called a **path** if its starting and ending vertices are different. It is called a **circuit** if the starting and ending vertices are the same.



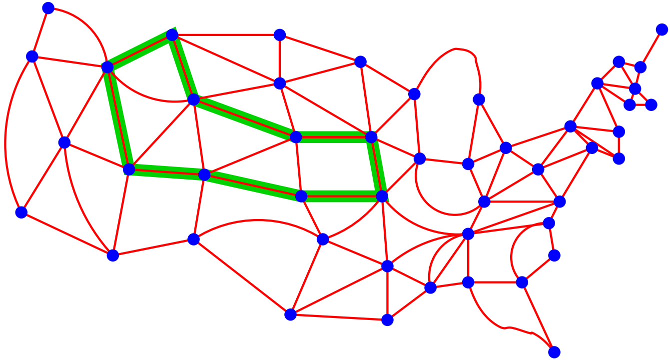
Trip #1 (a path)



Trip #2 (another path)



Trip #3 (a circuit)



A circuit without beginning or end

Connectedness

A graph is called **connected** if any two vertices can be linked by a path. Otherwise, it is **disconnected**.

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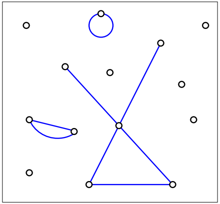
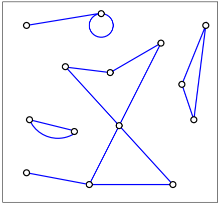
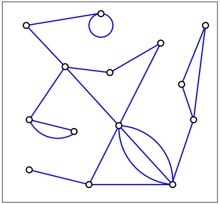


Isolated vertices
Disconnected



No isolated vertices
Disconnected



Connected

Bridges

Removing a single edge from a connected graph can make it disconnected. Such an edge is called a **bridge**.

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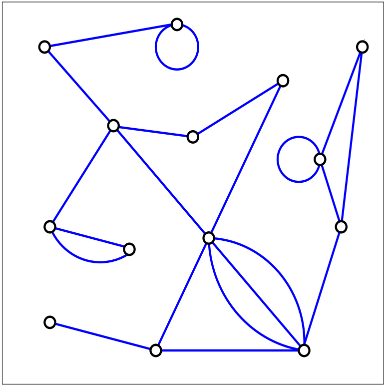

Bridges

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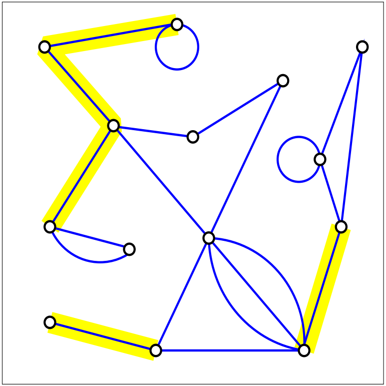
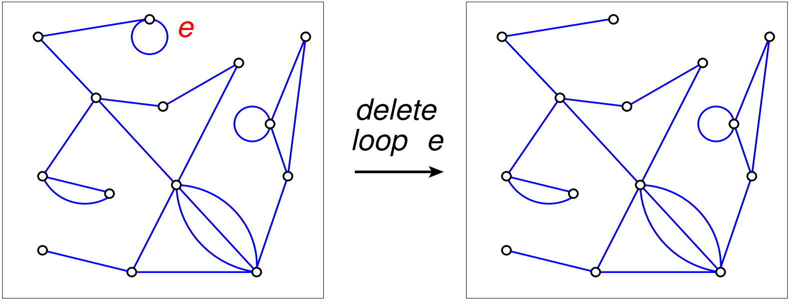


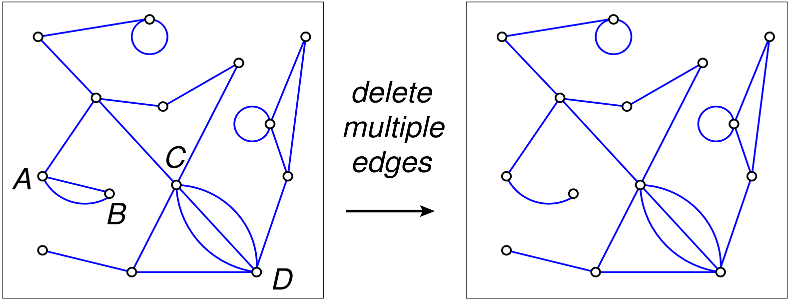
Bridges

Loops cannot be bridges, because removing a loop from a graph cannot make it disconnected.



Bridges

If two or more edges share both endpoints, then removing any one of them cannot make the graph disconnected. Therefore, none of those edges is a bridge.



Bridges

If two or more edges share both endpoints, then removing any one of them cannot make the graph disconnected. Therefore, none of those edges is a bridge.