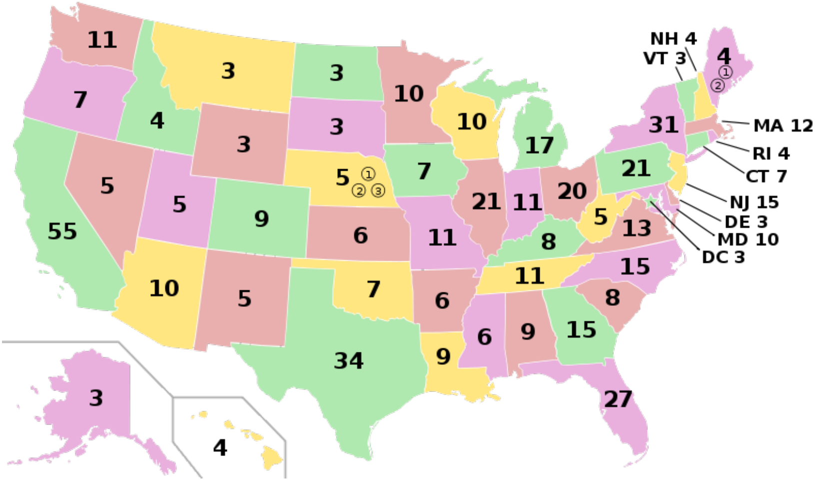
The Mathematics of Power (Chapter 2)

How can we use mathematics to study elections in which not all voters have the same number of votes?

- Electoral College (votes allocated to states by population)
- Parliamentary democracies (Great Britain, Canada, Italy, etc.; in fact, most democracies other than US)
- United Nations Security Council (five permanent members have veto power)
- Corporations (each share of stock is worth one vote)

The United States Electoral College



The United States Electoral College

As usual, we will focus on *mathematical* questions:

If a state has N electoral votes, how much influence does it have over the outcome of the election?

For example:

Kansas has twice as many electoral votes as North Dakota. Does Kansas have twice as much power?

These questions are independent of political questions (e.g., which are the swing states?)

Example: The UN Security Council

The UN Security Council consists of

- 5 permanent members
  (China, France, Russia, United Kingdom, USA)
- 10 rotating members
  (currently: Bosnia/Herzegovina, Brazil, Gabon, Lebanon, Nigeria, Colombia, Germany, India, Portugal, and South Africa)

Example: The UN Security Council

Article 27 of the UN Charter states:

1. *Each member of the Security Council shall have one vote.*
2. *Decisions of the Security Council . . . shall be made by an affirmative vote of nine members including the concurring votes of the permanent members.*

How much more actual power do the five permanent members have?

Weighted Voting Systems: Example 1

Let's look at a smaller example with just three voters (Huey, Dewey and Louie).

Huey	4 votes
Dewey	5 votes
Louie	5 votes
Total	**14 votes**
Majority	**8 votes**

Do Dewey and Louie have more power than Huey?

Weighted Voting Systems: Example 2

What if **two-thirds** of the votes are required to pass a motion?

Huey	4 votes	
Dewey	5 votes	
Louie	5 votes	
Total	**14 votes**	$\times \frac{2}{3} = 9\frac{1}{3}$
Required to pass	**10 votes**	

How much power does Huey really have now?

Weighted Voting Systems: Example 3

What if **three-quarters** of the votes are required to pass?

Huey	4 votes
Dewey	5 votes
Louie	5 votes
Total	**14 votes**
Required to pass	**11 votes**

$\times \frac{3}{4} = 10\frac{1}{2}$

Does this change make a difference?

Weighted Voting Systems: Example 3

What if **three-quarters** of the votes are required to pass?

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Total	**14 votes**
Required to pass	**11 votes**

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Does this change make a difference?

- In this system, any one voter can veto a measure he doesn't like.

Weighted Voting Systems: Example 4

Let's add one more voter.

Huey	4 votes
Dewey	5 votes
Louie	5 votes
Mom	26 votes
Total	**40 votes**
Majority	**21 votes**

Louie has one-eighth (5/40) of the votes. Does he therefore have one-eighth of the power?

Counting Votes vs. Measuring Power

Main Idea: Measuring power in a weighted voting system is more complex and subtle than merely finding the fraction of votes controlled by each voter.

How do we measure power using mathematics?

Specifically, how do we use math to tell when a voter has...

- dictatorial power?
- veto power?
- no power?
- more, equal or less power than another voter?

The Nassau County Board of Supervisors

The 1960s:

Nassau County, New York, is divided into six districts.

Each district elects one supervisor.

Each supervisor is allocated a number of votes proportional to the number of voters in his/her district.

The Nassau County Board of Supervisors

	District	Number of votes in 1964
(H1)	Hempstead #1	31
(H2)	Hempstead #2	31
(OB)	Oyster Bay	28
(NH)	North Hempstead	21
(LB)	Long Beach	2
(GC)	Glen Cove	2
	Total	**115**
	Majority	**58**

As a citizen of North Hempstead in 1964, what do you think about this arrangement? ★

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- Any two of H1, H2, OB can form a majority together.
- The other three districts **have zero power!**

Weighted Voting Systems

Weighted voting system: A voting system with N players, $P_1, P_2, \ldots, P_N$.

Weights: w_i = number of votes controlled by player P_i. (So the total number of votes is $V = w_1 + w_2 + \cdots + w_N$.)

Quota: q = number of votes necessary for a motion to pass. The quote can be a simple majority, or unanimity, or anything in between. That is,

$$V/2 < q \leq V.$$

(Why? Because $q \leq V/2$ would produce chaos, and $q > V$ would lead to gridlock.)

Weighted Voting Systems

Notation for a weighted voting system (WVS):

$$[q\,;\; w_1,\; w_2,\; \ldots,\; w_N].$$

- q is the quota.
- N is the number of players.
- The players' weights are $w_1,\ w_2,\ \ldots,\ w_N$.
- We'll always write the weights in decreasing order.
- The total number of votes is $V = w_1 + w_2 + \cdots + w_N$.

Weighted Voting Systems

Example: If Huey, Dewey and Louie have 4, 5, and 5 votes respectively (total 14), and a simple majority (8 votes) is needed to pass a motion, then the WVS is

$$[8;\ 5,\ 5,\ 4].$$

If instead a 2/3 majority (10 votes) is needed to pass a motion, then the WVS is

$$[10;\ 5,\ 5,\ 4].$$

Weighted Voting Systems

	District	Number of votes
(H1)	Hempstead #1	31
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	Total	**115**

If a majority (58 votes) is needed to pass a motion, then the WVS is

$$[58;\ 31,\ 31,\ 28,\ 21,\ 2,\ 2].$$

Weighted Voting Systems

Example: In the WVS [11; 6, 4, 4, 2, 1],

- There are $N = 5$ players P_1, P_2, P_3, P_4, P_5.
- The players control 6, 4, 4, 2, 1 votes respectively.
- A total of $q = 11$ votes are needed to pass a motion.
- Total number of votes:

$$V = 6 + 4 + 4 + 2 + 1 = \mathbf{17}.$$

- Therefore, the quota q must satisfy the inequalities

$$9 \leq q \leq 17.$$

Dictators

A **dictator** is a player who can pass a motion all by herself.

Player P_i is a dictator if $w_i \geq q$.

Example: [21; 26, 5, 5, 4] ("Huey, Dewey, Louie and Mom")

- Player P_1 is a dictator, because $w_1 = 26$ and $q = 21$.

Veto Power

A player has **veto power** if no motion can pass without his support.

Player P_i has veto power if $V - w_i < q$.

Example: [16; 8, **7**, 3, 2].

$$\begin{aligned} q &= 16 \\ V &= 8 + 7 + 3 + 2 = 20 \\ w_2 &= 7 \\ V - w_2 &= 13 \end{aligned}$$

So player P_2 (7 votes) has veto power.

Dictators vs. Veto Power

Every dictator has veto power.

But, not every player with veto power is necessarily a dictator.

For example, in the WVS [16; 8, 7, 3, 2], players P_1 and P_2 both have veto power, but neither is a dictator.

Here's an algebraic proof that every dictator has veto power.

Every Dictator Has Veto Power

- First of all, we know that $q > V/2$.

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- Now, **suppose player P_1 is a dictator**. That is,

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- Putting these two inequalities together, we get

$$w_1 \geq V - q$$

which says that P_1 **has veto power!**

Dummies

A **dummy** is a player with no power.

Example: [10; 5, 5, 4].

- Here P_3 is a dummy, because a motion passes if P_1 and P_2 both support it, but not otherwise.

Example: [58; 31, 31, 28, 21, 2, 2].

- Here, a motion passes if at least two of P_1, P_2, P_3 support it, but not otherwise.
- Therefore, P_4, P_5, and P_6 are all dummies.

How can we detect dummies mathematically?