Unbounded Matroids

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Matroids: Definition

A **matroid** M on ground set $E = [n]$ is a combinatorial structure that can be defined in many equivalent ways.

A **basis system** is a family $\mathcal{B}$ of subsets of E (called **bases**) such that:

- *Purity:* there is some $r \in \mathbb{N}$ such that $|B| = r$ for every $B \in \mathcal{B}$
  - r is called the **rank** of M.
- *Exchange:* For all $B, B' \in \mathcal{B}$:
  - $\forall e \in B \setminus B' : \quad \exists e' \in B' \setminus B : \quad B \setminus e \cup e' \in \mathcal{B}$
  - $\forall e \in B \setminus B' : \quad \exists e' \in B' \setminus B : \quad B' \setminus e' \cup e \in \mathcal{B}$

Matroids: Standard Examples

1. **Linear matroids.**
   - $E =$ set of vectors that span some vector space V
   - $\mathcal{B} =$ subsets of E that are bases for V
   - rank $= \dim V$
2. **Graphic matroids.**
   - $E =$ edge set of a connected graph G
   - $\mathcal{B} =$ spanning trees (maximal acyclic subsets of E)
   - rank $= |V(G)| - 1$

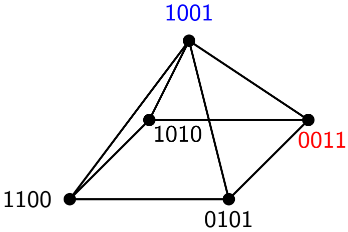
Matroid Polytopes

Every matroid M gives rise to a **matroid polytope**.

$$\text{basis } B \subseteq [n] \quad \rightsquigarrow \quad \text{characteristic vector } \chi_B \in \mathbb{R}^n$$

$$\text{matroid with basis system } \mathcal{B} \quad \rightsquigarrow \quad P_M = \text{conv}\{\chi_B \mid B \in \mathcal{B}\}$$

$$\mathcal{B} = \{12, 13, 14, 24, 34\}$$



Every edge of P_M corresponds to a basis exchange in $\mathcal{B}$, hence is parallel to a difference of two standard basis vectors.

Matroid Rank Functions

The **rank function** of a matroid M with basis system $\mathcal{B}$ on E is

$$\rho : 2^E \to \mathbb{N}, \qquad \rho(I) = \max_{B \in \mathcal{B}} |I \cap B|.$$

- Linear matroids: $\rho(I) = \dim \operatorname{span} I$
- Graphic matroids: $\rho(I) = |V(G)| - \#$ components of $G[I]$

General properties of rank functions:

- *bounded by cardinality*: $\rho(I) \leq |I|$
- *monotone*: $I \subseteq J \implies \rho(I) \leq \rho(J)$
- *submodular*: $\rho(I) + \rho(J) \geq \rho(I \cup J) + \rho(I \cap J)$.

Fact: Every function with these properties gives rise to a matroid basis system.

Matroid Rank Functions

Definition/Theorem: A **polytope** in $\mathbb{R}^n$ is (i) the convex hull of a finite point set. Equivalently, it is (ii) the bounded solution space of a finite system of linear equalities.

Type (i) description of P_M: uses its basis system.

Type (ii) description of P_M: uses its rank function:

$$P_M = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i \geq 0 & \forall i \in [n], \\ x_I \leq \rho(I) & \forall I \subseteq [n], \\ x_E = \rho(E) & \end{array} \right\}$$

where $x_I = \sum_{i \in I} x_i$.

Polymatroids

A **polymatroid rank function** is a function $\rho : 2^E \to \mathbb{R}_{\geq 0}$ that is

- *monotone*: $I \subseteq J \implies \rho(I) \leq \rho(J)$
- *submodular*: $\rho(I) + \rho(J) \geq \rho(I \cup J) + \rho(I \cap J)$.

A polymatroid rank function gives rise to a polytope called a **generalized permutahedron** ("genperm"):

$$P_\rho = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_I \leq \rho(I) \quad \forall I \subseteq E, \\ x_E = \rho(E) \end{array} \right\}$$

where $x_I = \sum_{i \in I} x_i$.

Matroids, Polymatroids, and Polytopes

- A polytope P is a genperm if and only if every edge is parallel to a difference of two standard basis vectors.
- Equivalently, the normal fan of P coarsens the braid fan.
- That is, the face of P maximizing some linear functional $\lambda(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ depends only on the relative *order* of $c_1, \dots, c_n$, not their specific values.
- Every matroid polytope is a genperm whose vertices are 0,1-vectors (not too hard).
- In fact, the converse is true (Gel'fand–Goresky–Macpherson–Serganova 1987; harder!)

Extended Generalized Permutahedra

- An **extended generalized permutahedron** (EGP) is a polyhedron P, not necessarily bounded, whose 1-dimensional faces (edges and rays) are all parallel to a difference of two standard basis vectors.
- Equivalently, the normal fan of P coarsens some subfan of the braid fan.
- That is, the face of P maximizing some bounded linear functional $\lambda(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ depends only on the relative *order* of $c_1, \ldots, c_n$, not their specific values.

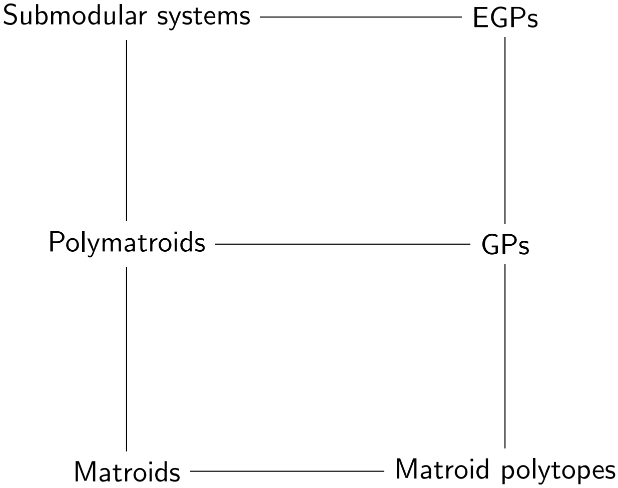
EGPs and Submodular Systems

A **submodular system** is a triple $M = (E, \mathcal{D}, \rho)$ such that:

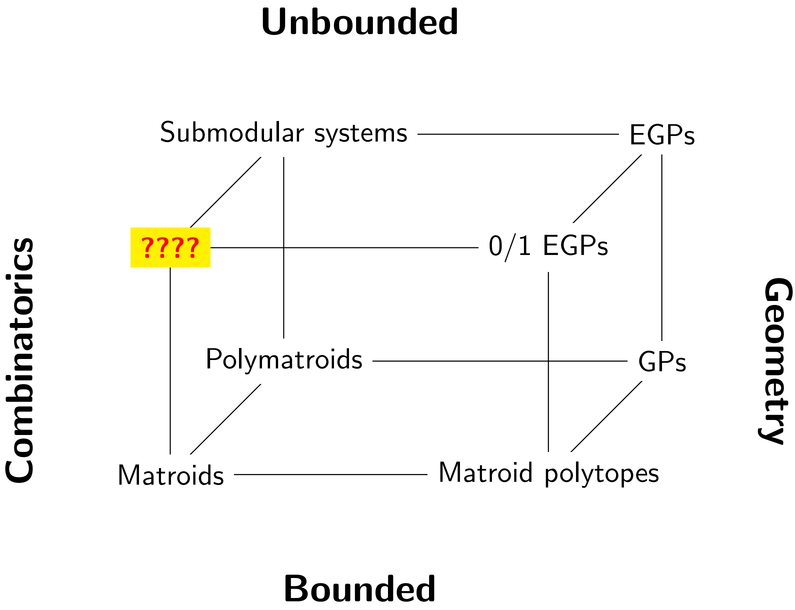
- $E = [n]$ is a finite set;
- $\mathcal{D}$ is a distributive sublattice of 2^E;
  - I.e., a family of subsets of E, containing $\emptyset$ and E, and closed under intersection and union
  - We typically assume $\mathcal{D}$ is *simple*, i.e., contains an element of every possible cardinality
- $\rho : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a monotone submodular function.

- Every submodular system M gives rise to an EGP P_M.
- Polymatroids are just submodular systems with $\mathcal{D} = 2^E$.
- The recession cone (the set of unbounded directions) is determined by $\mathcal{D}$ [details].

A Hierarchy

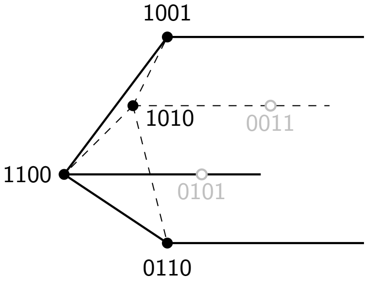

Submodular systems
EGPs
Polymatroids
GPs
Matroids
Matroid polytopes

A More Complete Hierarchy



Unbounded Matroids

This polyhedron is called (by us) the **stalactite**.



The corresponding submodular system is an **unbounded matroid**.

- Bases: $\{12, 13, 23, 14\}$ (*not* a matroid basis system!)
- $\mathcal{D} = \langle 12, 13, 23, 14 \rangle = 2^E \setminus \{4, 24, 34, 234\}$
- $\rho(A) = \min(A, |2|)$ for $A \subseteq [4]$

Unbounded Matroids

Definition

An **unbounded matroid** (or **D-matroid**) is a submodular system $(E, \mathcal{D}, \rho)$ satisfying the following conditions, for all $I, J \in \mathcal{D}$:

1. (*Integrality*) $\rho(I) \in \mathbb{Z}$;
2. (*Unit-increase*) If $J = I \cup \{e\}$, then $\rho(J) - \rho(I) \leq 1$.
   Equivalently, if $I \subseteq J$, then $\rho(J) - \rho(I) \leq |J| - |I|$.

Completing the Hierarchy

Theorem (BMRS 2021^+)

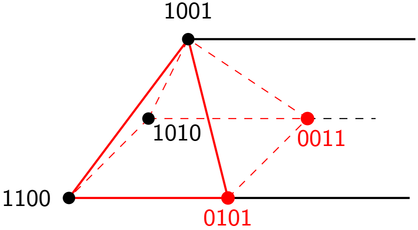
1. *A D-matroid is a matroid if and only if* $\mathcal{D} = 2^E$.
2. *A submodular system is a matroid if and only if it is both a polymatroid and a D-matroid.*
3. *The bijection between submodular systems and EGPs restricts to a bijection between D-matroids and 0,1-EGPs.*

Extensions of D-Matroids

Definition

Let $M = ([n], \mathcal{D}, \rho)$ be a submodular system. An **extension** of M is a D-matroid $N = (E, \mathcal{D}', \sigma)$ such that $\mathcal{D} \subseteq \mathcal{D}'$ and $\sigma|_{\mathcal{D}} = \rho$.

In this case $P_N \subseteq P_M$ and $V(P_N) \supseteq V(P_M)$. We say that P_N is a **HIVE polyhedron**[1] of P_M.



[1] "hull-internal, vertex-external"

Extensions of D-Matroids

- When do extensions exist?
- Does every 0/1-EGP have a HIVE polytope? Equivalently, can every D-matroid be extended to a matroid?
- If so, is there a "canonical" matroid extension of any given D-matroid?

Generous Extensions

Let $(E, \mathcal{D}, \rho)$ be a D-matroid. Let $a \in E$ such that $\{a\} \notin \mathcal{D}$. Let $\mathcal{D}[a]$ be the distributive sublattice of 2^E generated by $\mathcal{D}$ and $\{a\}$.

For $J \subseteq E$, define $\sup_{\mathcal{D}}(J) = \bigcap_{\substack{K \in \mathcal{D} \\ K \supseteq J}} K$.

The **generous extension** of ρ to $\mathcal{D}[a]$ is the function $\rho_a : \mathcal{D}[a] \to \mathbb{N}$ defined by

$$\rho_a(J) = \begin{cases} \rho(J) & \text{if } J \in \mathcal{D}, \\ \rho(J - a) & \text{if } J \notin \mathcal{D} \text{ and } \rho(J - a) = \rho(\sup_{\mathcal{D}}(J)), \\ \rho(J - a) + 1 & \text{if } J \notin \mathcal{D} \text{ and } \rho(J - a) < \rho(\sup_{\mathcal{D}}(J)). \end{cases}$$

Theorem (BMRS 2021$^+$)

Let $M = (E, \mathcal{D}, \rho)$ be a D-matroid.

1. *The generous extension ρ_a is a D-matroid rank function (monotone, unit-increase, and submodular).*
2. *ρ_a dominates all other extensions. That is, if $N = (E, \mathcal{D}[a], \sigma)$ is any extension of M, then $\rho_a(J) \geq \sigma(J)$ for all J.*
3. *The iterated generous matroid extension $\hat{\rho}$ of ρ to 2^E is independent of the order of iteration, and dominates every other matroid extension.*
4. *The foregoing is true if 2^E is replaced with any lattice $\mathcal{D}'$ between $\mathcal{D}$ and 2^E.*

Corollary

Every 0,1-EGP P contains a unique maximal matroid polytope $\hat{P}$. Moreover, $\hat{P} = Q + R(P)$, where $R(P)$ is the recession cone.

More Questions

- Which pure set systems arise as unbounded matroids?
- Is the maximal matroid subpolytope $\hat{P}$ equal to the convex hull of the 0/1-vectors in P?
- What do the normal fans of D-matroid polyhedra look like? (The supports of normal fans of submodular systems are essentially preposets.)
- What about non-generous extensions?
- What can you say about the poset of all HIVE polyhedra/polytopes of P ordered by inclusion?
- Do other matroid axiomatizations (bases, circuits, closure operator, greedy algorithm, ...) have reasonable D-matroid analogues? (Some do for general submodular systems.)
- Are D-matroid complexes shellable? (José thinks yes; he and Ignacio are working on it.)
- Applications in combinatorial optimization?

Thanks!

The Details

Proposition

Let $M = ([n], \mathcal{D}, \rho)$ be a submodular system. A linear functional $\lambda(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ is bounded on P_M if and only if $\mathcal{D}$ contains every $J \subseteq [n]$ such that $c_j \geq c_i$ for all $j \in J$ and $i \notin J$.

Example

Let $\lambda(\mathbf{x}) = 4x_1 - x_2 - 2x_3 + 3x_4 - x_5$, so that

$$c_1 > c_4 > c_2 = c_5 > c_3.$$

Then λ is bounded on P_M if and only if $\mathcal{D}$ contains each of

$$\emptyset,\ 1,\ 14,\ 142,\ 145,\ 1425,\ 14253.$$