Ehrhart Theory for Paving Matroids

Mohsen Aliabadi (Iowa State)
Derek Hanely (U Kentucky)
Jeremy L. Martin (KU)
Daniel McGinnis (Iowa State)
Dane Miyata (U Oregon)
George D. Nasr (U Oregon)
Andrés R. Vindas-Meléndez (UC Berkeley)
Mei Yin (U Denver)

KU Combinatorics Seminar
October 22, 2021

Matroids: Definition

A **matroid** M on ground set $E = [n]$ is a combinatorial structure that can be defined in many equivalent ways.

Basis system: family $\mathcal{B}$ of subsets of E (called **bases**) such that:

- *Purity:* there is some $r \in \mathbb{N}$ such that $|B| = r$ for every $B \in \mathcal{B}$
  - r is called the **rank** of M.
- *Exchange:* For all $B, B' \in \mathcal{B}$:
  - $\forall e \in B \setminus B' : \quad \exists e' \in B' \setminus B : \quad B \setminus e \cup e' \in \mathcal{B}$
  - $\forall e \in B \setminus B' : \quad \exists e' \in B' \setminus B : \quad B' \cup e \setminus e' \in \mathcal{B}$

Subsets of bases are called **independent sets**.

- The independent sets form a simplicial complex Δ.
- *Donation:* $I, J \in \Delta,\ |I| < |J| \implies \exists e \in J \setminus I :\ I \cup e \in \Delta$.

Matroid Basics

Let M be a matroid on ground set $E = [n]$ with basis system $\mathcal{B}$.

Dual: matroid M^* with basis system $\mathcal{B}^* = \{E \setminus B : \; B \in \mathcal{B}\}$

Rank function: $\rho : 2^E \to \mathbb{N}$ defined by $\rho(A) = \max_{B \in \mathcal{B}} |A \cap B|$

Circuits of M: minimal *dependent* sets = smallest sets not contained in any basis of M

Cocircuits of M = circuits of M^* = smallest sets intersecting every basis of M nontrivially

Fact

Basis system, independence complex, rank function, circuit system, and cocircuit system are "cryptomorphic" (all contain equivalent information and characterize a matroid).

Paving and Sparse Paving Matroids

Let M be a matroid of rank r. Then every circuit of M has cardinality at most $r+1$ (because $|A| \geq r+1 \implies A$ dependent)

Fact

The only matroid on $[n]$ with no circuits of size $< r+1$ is the **uniform matroid** $U_r(n)$, with basis system $\binom{[n]}{r}$.

Definition

M is a **paving matroid** if it has no circuits of size $< r$.
M is **sparse paving** if M and M^* are both paving.

Example

Projective plane matroids ($E = (\mathbb{F}_q^3 \setminus \{0\})/\mathbb{F}_q^\times$) are paving, but not sparse paving (except $q = 2$)

Conjecture

Almost all matroids are paving matroids.

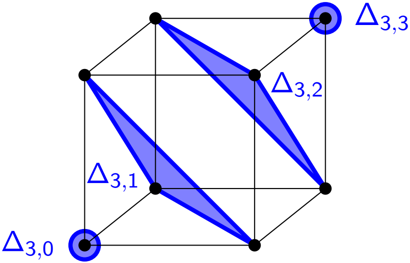
Matroid Polytopes

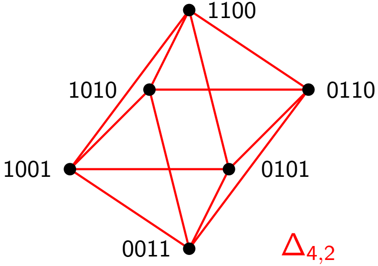
Every matroid M on $[n]$ has a **matroid base polytope**:

$$P_M = \text{conv}\{\chi_B \mid B \in \mathcal{B}\}$$

The matroid base polytope of $U_r(n)$ is the **hypersimplex:**

$$\Delta_{n,r} = \text{conv}\{\text{vectors in } \mathbb{R}^n \text{ with } r \text{ 1's and } n-r \text{ 0's}\}$$





- $\Delta_{n,0} \cong \Delta_{n,n} =$ point; $\Delta_{n,1} \cong \Delta_{n,n-1} =$ simplex
- $\Delta_{n,k} \cong \Delta_{n,n-k}$ (note: $P_M \cong P_{M^*}$ in general)

Ehrhart Theory

Let $P \subseteq \mathbb{R}^n$ be a polytope and $k \in \mathbb{N}$.

kth dilate of P: $\{kx : \ x \in P\}$
Ehrhart function: $\mathrm{ehr}_P(k) = \#(kP \cap \mathbb{Z}^n)$

P	$\mathrm{ehr}_P(k)$
Unit cube $[0,1]^n$	$(k+1)^n$
Simplex $\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n)$	$\#\{(x_1, \ldots, x_n) : \ x_i \leq 0, \ 0 \leq \sum x_i \leq k\}$ $= \binom{k+n}{n} = \frac{(k+n)(k+n-1)\cdots(k+1)}{n!}$
Line segment $[0, 1/2]$	$\begin{cases} \frac{k+1}{2} & \text{k odd} \\ \frac{k+2}{2} & \text{k even} \end{cases}$

Ehrhart Theory

Ehrhart's Theorem: (1) If P is a **lattice polytope** (vertices in $\mathbb{Z}^n$) then $\text{ehr}_P(k)$ is a **polynomial** in k.

- Degree of $\text{ehr}_P(k) = \dim P$
- Leading coefficient = volume

(2) if P has **vertices in $\mathbb{Q}^n$** then $\text{ehr}_P(k)$ is a **quasipolynomial:**

$$\text{ehr}_P(k) = \begin{cases} f_0(k) & \text{if } k \equiv 0 \mod p \\ \vdots & \vdots \\ f_{p-1}(k) & \text{if } k \equiv p-1 \mod p \end{cases}$$

where $f_0, \ldots, f_{p-1}$ are polynomials in k; p is the *period* of P.

- When are the coefficients of the Ehrhart polynomial positive?
- What are the Ehrhart functions of matroid polytopes?

Ehrhart Theory of Matroid Polytopes

Volume of a matroid polytope: Ardila–Benedetti–Doker 2010 (uses toric varieties, other ingredients)

Ehrhart polynomial of a hypersimplex: Katzman 2005 (commutative algebra context):

$$\mathrm{ehr}_{\Delta_{n,r}}(k) = \sum_{j=0}^{r-1} (-1)^j \binom{n}{j} \binom{(r-j)k - j + n - 1}{n-1}$$

Conjecture (De Loera–Haws–Köppe 2009) Matroid base polytopes are Ehrhart positive.

Theorem (Ferroni 2021)

- Positivity conjecture **fails** for certain "sparse paving matroids"
- Positivity holds for hypersimplices and "minimal matroids"

A Recharacterization of Paving Matroids

Theorem (MA-DH-JLM-DMcG-DM-GDN-ARVM-MY 2021$^+$) Let $r \leq n$. Let $\mathcal{H} \subset 2^{[n]}$ be a set family satisfying the following conditions:

1. $|H| \geq r$ for all $H \in \mathcal{H}$.
2. $\bigcup_{H \in \mathcal{H}} \binom{H}{r} \neq \binom{[n]}{r}$ (to avoid trivialities).
3. If $H, H' \in \mathcal{H}$ with $H \neq H'$, then $|H \cap H'| \leq r - 2$.

Then the family

$$\mathcal{B}(n, r, \mathcal{H}) = \left\{ B \in \binom{[n]}{r} \;\middle|\; B \not\subseteq H \quad \forall H \in \mathcal{H} \right\}$$

is a matroid basis system for a matroid $M(n, r, \mathcal{H})$. The elements of $\mathcal{H}$ are **hyperplanes.**

Ehrhart Polynomials of Paving Matroids

Idea: For $M = M(n, r, \mathcal{H})$ a paving matroid, construct P_M from $\Delta_{n,r}$ by 'cutting off corners".

Example: If $\mathcal{H} = \{H\}$: Delete vertices of $\Delta_{n,r}$ with all r of their 1-coordinates in H; keep vertices with a 1-coordinate outside H. Equivalently, impose either of the equivalent inequalities

$$\sum_{i \in H} x_i \leq r - 1 \qquad \text{or} \qquad \sum_{i \in [n] \setminus H} x_i \geq 1$$

Then $P \cup Q = \Delta_{n,r}$ and $P \cap Q = R$, where

$$Q = \text{conv}\{v \in V(H) \mid \textstyle\sum_{i \in H} v_i = r - 1 \text{ or } r\},$$
$$R = \text{conv}\{v \in V(H) \mid \textstyle\sum_{i \in H} v_i = r - 1\}$$

so $\boxed{\text{ehr}_P = \text{ehr}_{\Delta_{n,r}} - \text{ehr}_Q + \text{ehr}_R\,.}$

Ehrhart Polynomials of Paving Matroids

Thus, we need to understand the Ehrhart polynomials of

$$Q = \text{conv}\{v \in V(H) \mid \sum_{i \in H} v_i = r - 1 \text{ or } r\},$$

$$R = \text{conv}\{v \in V(H) \mid \sum_{i \in H} v_i = r - 1\}.$$

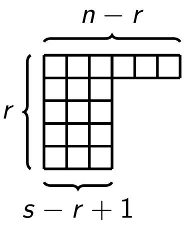
- R is the easy one: it is $\Delta_{|H|,r-1} \times \Delta_{n-|H|,1}$.
- Q is the subtle one: it is a thing called (by us) a **panhandle matroid**, which is an instance of a **lattice path matroid**

Lattice Path Matroids

Use the chalk, Jeremy.

Panhandle Matroids

A **panhandle matroid** $\mathrm{Pan}(n, r, s)$ is a lattice path matroid for a partition shaped like this:



Key observation: The polytope

$$Q = \mathrm{conv}\{v \in V(H) \mid \sum_{i \in H} v_i = r - 1 \text{ or } r\}$$

is the base polytope of the panhandle matroid $\mathrm{Pan}(n, r, |H|)$.

Ehrhart Polynomials of Paving Matroids

Example Again: If $M(n, r, \{H\})$ is a paving matroid with one hyperplane, then

$$\begin{aligned} \mathrm{ehr}_{P_M} &= \mathrm{ehr}_{\Delta_{n,r}} - \mathrm{ehr}_Q + \mathrm{ehr}_R \\ &= \underbrace{\mathrm{ehr}_{\Delta_{n,r}}}_{\text{Katzman}} - \mathrm{ehr}_{\mathrm{Pan}(n,r,|H|)} + (\mathrm{ehr}_{\Delta_{|H|,r-1}})(\mathrm{ehr}_{\Delta_{n-|H|,1}}) \end{aligned}$$

Good News: The corners don't overlap!

So for every paving matroid $M = M(n, r, \mathcal{H})$:

$$\mathrm{ehr}_{P_M} = \mathrm{ehr}_{\Delta_{n,r}} + \sum_{H \in \mathcal{H}} \left(-\mathrm{ehr}_{\mathrm{Pan}(n,r,|H|)} + (\mathrm{ehr}_{\Delta_{|H|,r-1}})(\mathrm{ehr}_{\Delta_{n-|H|,1}}) \right)$$

Ehrhart Polynomials of Panhandle Matroids

Conjecture: Panhandle matroids are Ehrhart positive.

We can do a whole lot of algebra to boil this down to the conjecture that

$$\xi(k,\ell,m,r,s) = \sum_{i=0}^{r} (-1)^i \binom{s}{i} \binom{k+r-i-1}{k-1} \times e_{s-\ell-m}[-i+1, s-1-\ell-i] e_{\ell-k+m}[s-\ell-i, s-1-i]$$

is positive for all k, ℓ, m, r, s in appropriate ranges.

We are sure that $\xi(k,\ell,m,r,s)$ is the number of labeled "chain gangs" of a certain kind. . .