Hopf Monoids of Ordered Simplicial Complexes

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Hopf Algebras and Hopf Monoids

Hopf algebras: framework for studying combinatorial objects that can be merged (product) and broken (coproduct).

- Ex.: graphs, posets, matroids, simplicial complexes, (quasi)symmetric functions, rooted trees, ...
- Objects are unlabeled $\implies$ loss of information

Hopf monoids: record product and coproduct of labeled objects

- Hopf algebra: $\mathcal{G} = \bigoplus_{n \geqslant 0} \mathbb{k}\{\text{graphs on } n \text{ vertices}\}$
- Hopf monoid: $\mathbf{G}[S] = \mathbb{k}\{\text{graphs on vertex set } S\}$

We want to study: simplicial complexes whose vertices are not just labeled, but **ordered.**

Hopf Monoids of Ordered Labeled Things

We want to study Hopf monoids of simplicial complexes) whose vertices are **ordered.**

- Some are egalitarian with respect to orders (prototype: Hopf monoid **Mat** of matroid complexes)
- Some derive structure from a specific order (shifted complexes, broken-circuit complexes)

- Technical problems with extending **Mat**
- Introducing an ordering helps overcome these problems

Hopf Monoids (briefly!)

Hopf monoid in set species: family $\{\mathbf{h}[I] \mid I \text{ finite}\}$, with maps

$$\mu_{I_1,\dots,I_n} : \mathbf{h}[I_1] \times \cdots \times \mathbf{h}[I_n] \to \mathbf{h}[I_1 \sqcup \cdots \sqcup I_n] \quad (\textbf{product})$$
$$\Delta_{I_1,\dots,I_n} : \mathbf{h}[I_1 \sqcup \cdots \sqcup I_n] \to \mathbf{h}[I_1] \times \cdots \times \mathbf{h}[I_n] \quad (\textbf{coproduct})$$

satisfying compatibility, associativity, coassociativity, ...

Shorthand notation: $\mu_{I,J}(a, b) = a \cdot b$; $\Delta_{I,J}(a) = a|_I \times a/_I$

Hopf monoid in vector species: $\mathbf{H}[I] = \Bbbk\mathbf{h}[I]$; replace $\times$ by $\otimes$

- **Linearization** of $\mathbf{h}$: extend set-theoretic μ and Δ linearly
- Not every interesting vector Hopf monoid is a linearization!

The Hopf Monoid of Matroids

$\mathbf{Mat}[I]$ = linear combinations of matroids on ground set I
(equivalently, matroid independence complexes)

Product: direct sum of matroids / join of complexes
commutative: $M \cdot N = N \cdot M$

Coproduct: restriction/contraction: $\Delta_{I,J}(M) = M|I \otimes M/I$
not *cocommutative:* $M|I \neq M/J$

Goal: Adapt to pure ordered simplicial complexes.

Irksome Fact: **Mat** is the unique largest species of pure simplicial complexes **SC** with this Hopf structure
(because Δ matroidal $\iff$ $\Delta|I$ pure for all I)

Interlude: Another Characterization of Matroids

Theorem (Castillo–JLM–Samper 2020^+)

Let Γ be a simplicial complex on ground set E. Then Γ is a matroid complex if and only if it is **link-invariant:** *for every $X \subseteq E$ and every facets $\tau, \sigma \in \Gamma|_X$ we have*

$$\mathrm{link}_\Gamma(\sigma) = \mathrm{link}_\Gamma(\tau).$$

(Has anyone seen this before?)

The Hopf Monoid of Linear Orders

$\mathbf{L}[I] = \mathbb{k}\{\text{linear orders on } I\}$
ex.: $\mathbf{L}[\{2,5,6\}] = \mathbb{k}\{256, 265, 526, 562, 625, 652\}$

Product: "concatenate" $14 \cdot 32 = 1432$
not commutative

Coproduct: "split" $\Delta_{I,J}(1324) = 14 \otimes 32$
cocommutative

This is not the Hopf structure we are looking for.

The Dual Hopf Monoid of Linear Orders

$\mathbf{L}^*[I] = \Bbbk\{$linear orders on I

Product: $$\mu^*_{I,J}(u \otimes v) = \sum_{w \in \text{Shuffle}(u,v)} w$$

Ex: $14 * 32 = 1432 + 1342 + 1324 + 3142 + 3124 + 3214 = 32 * 14$

Coproduct: $$\Delta^*_{I,J}(w) = \begin{cases} w|_I \otimes w|_J & \text{if } w = w|_I \cdot w|_J \\ 0 & \text{otherwise.} \end{cases}$$

- This is a better product for ordered simplicial complexes (since join is naturally commutative)
- Turns out to have other good features (stay tuned)

Ordered simplicial complexes and Hopf classes

Ordered simplicial complex: triple (w, Γ, I) where $\Gamma =$ pure simplicial complex on I and $w \in \boldsymbol{\ell}[I]$

Initial restriction/contraction: for $A \subseteq I$ an initial segment of w,

$$(w, \Gamma)|A = (w|_A, \Gamma|A, A), \qquad (w, \Gamma)/A = (w|_{I \setminus A}, \mathrm{link}_\Gamma(\phi), I \setminus A)$$

where $\phi = w$-lex-minimal facet of $\Gamma|A$

Hopf class: collection of ordered simplicial complexes closed under initial restriction, initial contraction, and ordered join (i.e., simplicial join with any shuffle of orders)

- Largest Hopf class: **prefix-pure** complexes (all restrictions to initial segments are pure).

  Much more general than matroids!

Hopf classes and Hopf monoids

Theorem (Castillo–JLM–Samper 2020^+)

Every Hopf class H *gives rise to a commutative Hopf monoid* $\mathbf{H}$ *in vector species.*

- $\mathbf{H} \subset \mathbf{L}^* \times \mathbf{SC}$ as vector species (and as Hopf monoids if elements of H are matroids)

Examples:

- Joins of shifted complexes
- Prefix-pure + lex-shellable
- Quasi-matroidal classes (Samper)
- Gale truncations of shifted complexes
- Color-shifted complexes
- Broken-circuit complexes
- ...

Matroids and generalized permutahedra

$M \in \mathbf{Mat}[I] \rightsquigarrow$ **base polytope** $P_M = \mathrm{conv}\{\chi_B \mid B \in \mathcal{B}_M\} \subset \mathbb{R}^I$

Matroid base polytopes are **generalized permutahedra:** edges parallel to $\mathbf{e}_i - \mathbf{e}_j \iff$ normal fan coarsens braid arrangement

Theorem (Gel'fand–Goresky–Macpherson–Serganova 1987)

Matroid base polytopes are precisely the genperms with 0/1 coefficients.

- Genperms **GP** form a Hopf monoid [Aguiar–Ardila]
- $M \mapsto P_M$ is a monoid monomorphism $\mathbf{Mat} \to \mathbf{GP}$
- Ordered matroids $\mathbf{L}^* \times \mathbf{Mat} \to \mathbf{L}^* \times \mathbf{GP}$

  (genperms with ordering on coordinates)

Antipodes

Antipode on a vector Hopf monoid $\mathbf{H}$:

- Generalizes inversion in groups
- Applications: character group, combinatorial reciprocity, etc.

Takeuchi formula (always valid, seldom optimal):

$$\mathbf{s}(w) = \sum_{\Phi=\Phi_1|\cdots|\Phi_k\models[n]} (-1)^k \mu_\Phi(\Delta_\Phi(w))$$

Cancellation- and multiplicity-free formulas:

In $\mathbf{L}$ and $\mathbf{L}^*$ $\quad \mathbf{s}(w) = (-1)^{|w|} w^{\mathrm{rev}}$

In $\mathbf{GP}$ $\quad \mathbf{s}(\mathfrak{p}) = \sum_{\text{faces } \mathfrak{q}\subset\mathfrak{p}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}$ (Aguiar–Ardila)

(Idea: coefficients are Euler characteristics)

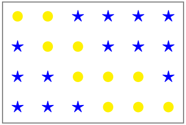
The Antipode on $\mathbf{L}^* \times$ OGP

Antipodes do *not* play nicely with Hadamard product. Yet. . .

Theorem (Castillo–JLM–Samper 2020$^+$)

The antipode in $\mathbf{L}^* \times \mathbf{GP}$ *(in fact, in* $\mathbf{OGP}^+$*)*[1] *is given by [a very involved formula that is cancellation-free and multiplicity-free].*

- Proof uses Ardila–Aguiar topological method
- Antipode is *local*: $u \otimes \mathfrak{q} \in \mathrm{supp}(\mathbf{s}(w \otimes \mathfrak{p})) \implies \mathfrak{q}$ contains the vertex of $\mathfrak{p}$ maximized by w (unlike $\mathbf{L} \times \mathbf{GP}$)
- Wild card: Euler characteristics of **Scrope complexes.**



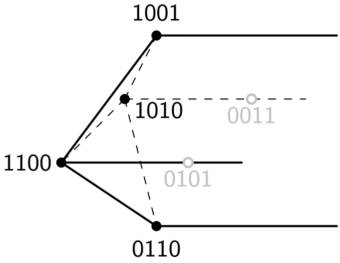
$\rightsquigarrow \langle 3456,\ 1456,\ 126,\ 123 \rangle$

- Every Scrope complex is a homotopy ball or sphere. (Which??)

1 A larger Hopf monoid that includes *unbounded* genperms.

New Directions: Unbounded Matroids

$\mathfrak{p}$ = OIEGP (possibly unbounded genperm with 0/1 coordinates)
$\Upsilon(\mathfrak{p}) = \langle$supports of vertices of $\mathfrak{p}\rangle$



$\Upsilon(\mathfrak{p}) = \langle 12, 13, 23, 14 \rangle$
(*not* a matroid complex!)

Theorem (Castillo–JLM–Samper 2020$^+$)
$\{(w, \Upsilon(\mathfrak{p}), I) :\ \mathfrak{p} \subset \mathbb{R}^I$ *OIEGP, w maximized on* $\mathfrak{p}\}$ *is a Hopf class.*

These "unbounded matroids" have a **lot** of structure
(Jonah Berggren–JLM–Ignacio Rojas–Samper, in progress)

Thank you!

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Okay, You Asked For It

Theorem (Castillo, JLM, Samper 2021$^+$)

The antipode in $\mathbf{OGP}_+$ *is given by the following extremely impressive formula (which is multiplicity- and cancellation-free):*

$$\mathbf{s}(w \otimes \mathfrak{p}) = \sum_{\substack{u \in \boldsymbol{\ell}_{\mathfrak{p}}[I],\ \mathfrak{q} \subset \mathfrak{p}: \\ \mathrm{D}_u = \mathrm{N}_{w,Q},\ \mathsf{C}_{\mathrm{W}} \cap \mathsf{C}_Q^\circ \neq \emptyset}} (-1)^{1+\mathsf{des}(u^{-1}w)} u \otimes \mathfrak{q} + \sum_{\substack{u \in \boldsymbol{\ell}_{\mathfrak{p}}[I],\ \mathfrak{q} \subset \mathfrak{p}: \\ \mathrm{D}_u \in \partial \mathsf{C}_Q,\ \mathsf{C}_{\mathrm{W}} \cap \mathsf{C}_Q^\circ \neq \emptyset}} (-1)^{1+\mathsf{des}(u^{-1}w)} \tilde{\chi}(\check{\mathcal{G}}) u \otimes \mathfrak{q}$$

where

$$\begin{aligned} \boldsymbol{\ell}_{\mathfrak{p}}[I] &= \{w \in \boldsymbol{\ell}[I] : \textstyle\sum_i w_i x_i \text{ bounded on } \mathfrak{p}\} \\ \mathrm{D}_u,\ \mathrm{N}_{w,Q} &= \text{set compositions} \\ \mathsf{C}_{\mathrm{W}},\ \mathsf{C}_Q^\circ &= \text{subfans of braid arrangement} \\ \check{\mathcal{G}} &= \text{Scrope complex depending on } Q, w, u \end{aligned}$$