Math 824, Fall 2010
Problem Set #4
Due date: Friday 11/5/10

Problem #1 (Stanley, EC1, 3.45) Prove the *q-binomial theorem*:

$$\prod_{k=0}^{n-1}(x-q^k) \;=\; \sum_{k=0}^{n} \binom{\mathbf{n}}{\mathbf{k}} (-1)^k q^{\binom{k}{2}} x^{n-k}.$$

Here $\binom{\mathbf{n}}{\mathbf{k}}$ denotes the *q-binomial coefficient*:

$$\binom{\mathbf{n}}{\mathbf{k}} \;=\; \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})};$$

see Stanley, EC1, pp. 26–28, or Aigner [who uses the notation $\binom{n}{k}_q$ and calls them "Gaussian coefficients"], pp. 69, 79, 94. You may, with appropriate citation, use identities such as (17b) on p. 26 of Stanley.

(Hint: Let $V = \mathbb{F}_q^n$ and let X be a vector space over $\mathbb{F}_q$ with x elements. Count the number of one-to-one linear transformations $V \to X$ in two ways.) Derive the ordinary binomial theorem as a corollary.

Problem #2 (Stanley, EC1, Supp. 4) Let P be a finite poset, and let μ be the Möbius function of $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. Suppose that P has a fixed-point-free automorphism $\sigma : P \to P$ of prime order p; that is, $\sigma(x) \neq x$ and $\sigma^p(x) = x$ for all $x \in P$. Prove that $\mu(\hat{0}, \hat{1}) \cong -1 \pmod{p}$. What does this say in the case that $\hat{P} = \Pi_p$?

Problem #3 (Stanley, HA, 2.5) Let K be a field, let G be a graph on n vertices, and let $\mathcal{B}_G = \mathscr{B}_n \cup \mathcal{A}_G$; that is, $\mathcal{B}$ consists of the coordinate hyperplanes in K^n together with the hyperplanes $x_i = x_j$ for all edges ij of G. Calculate $\chi_{\mathcal{B}_G}(k)$ in terms of $\chi_{\mathcal{A}_G}(k)$.

Problem #4 Consider the permutation action of the symmetric group $\mathfrak{S}_4$ on the vertices of the complete graph K_4, whose corresponding representation is the defining representation ρ_{def} (let's say over $\mathbb{C}$). Let σ be the 3-dimensional representation corresponding to the action of $\mathfrak{S}_4$ on pairs of opposite edges of K_4.

(#4a) Compute the character of σ.

(#4b) Explicitly describe all G-equivariant linear transformations $\phi : \rho_{\text{def}} \to \sigma$. (Hint: Schur's lemma should be useful.)

Problem #5 Recall that the *alternating group* $\mathfrak{A}_n$ consists of the $n!/2$ even permutations in $\mathfrak{S}_n$, that is, those with an even number of even-length cycles.

(#5a) Show that the conjugacy classes in $\mathfrak{A}_4$ are not simply the conjugacy classes in $\mathfrak{S}_4$. (Hint: Consider the possibilities for the dimensions of the irreducible characters of $\mathfrak{A}_4$.)

(#5b) Determine the conjugacy classes in $\mathfrak{A}_4$, and the complete list of irreducible characters.

(#5c) Use this information to determine $[\mathfrak{A}_4, \mathfrak{A}_4]$ without actually computing any commutators.