Math 824, Fall 2010
Problem Set #2
Due date: Friday 9/17/10

Problem #1 Prove that the partition lattice Π_n is a geometric lattice. (Hint: Construct a collection of vectors $S = \{s_{ij} \mid 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$ such that $L(S) \cong \Pi_n$.)

Problem #2 Let $\mathbb{F}$ be a field, let $n \in \mathbb{N}$, and let S be a finite subset of the vector space $\mathbb{F}^n$. Recall the definitions of the posets $L(S)$ and $L^{\text{aff}}(S)$ (see §1.8 of the lecture notes). For $s = (s_1, \dots, s_n) \in S$, let $\hat{s} = (1, s_1, \dots, s_n) \in \mathbb{F}^{n+1}$, and let $\hat{S} = \{\hat{s} \mid s \in S\}$. Prove that $L(\hat{S}) = L^{\text{aff}}(S)$.

Problem #3 Determine, with proof, all pairs of integers $k \leq n$ such that there exists a graph G with $M(G) \cong U_k(n)$. (Recall that $U_k(n)$ is the matroid on $E = [n]$ such that every subset of E of cardinality k is a basis.)

Problem #4 Prove that the two forms of the basis exchange condition are equivalent. That is, if $\mathscr{B}$ is a family of subsets of a finite set E, all of the same cardinality, then prove that

$$\text{for every } e \in B \setminus B', \text{ there exists } e' \in B' \setminus B \text{ such that } B \setminus \{e\} \cup \{e'\} \in \mathscr{B}$$

if and only if

$$\text{for every } f \in B \setminus B', \text{ there exists } f' \in B' \setminus B \text{ such that } B' \setminus \{f'\} \cup \{f\} \in \mathscr{B}.$$

(Hint: Consider $|B \setminus B'|$.)

Problem #5 Let M be a matroid on ground set E with rank function r. Let M^* be the dual matroid to M, and let r^* be its rank function. Find a formula for r^* in terms of r.

Problem #6 Let E be a finite set and let $\mathscr{I}$ be a simplicial complex on E (that is, a family of subsets such that if $A \in \mathscr{I}$ and $B \subseteq A$, then $B \in \mathscr{I}$). Let $w : E \to \mathbb{R}_{\geq 0}$ be any weight function. For $A \subseteq E$, define $w(A) = \sum_{e \in A} w(e)$. Consider the problem of maximizing $w(A)$ over all maximal† elements $A \in \mathscr{I}$ (also known as *facets* of $\mathscr{I}$). A naive approach to try to produce such a set A, which may or may not work for a given $\mathscr{I}$ and w, is the following *greedy algorithm*:

(1) Let $A = \emptyset$.
(2) If A is a facet of $\mathscr{I}$, stop.
Otherwise, find $e \in E \setminus A$ of maximal weight such that $A \cup \{e\} \in \mathscr{I}$, and replace A with $A \cup \{e\}$.
(3) Repeat step 2 until A is a facet of $\mathscr{I}$.

(#6a) Construct a simplicial complex and a weight function for which this algorithm does not produce a facet of maximal weight. (Hint: The smallest example has $|E| = 3$.)

(#6b) Prove that the following two conditions are equivalent:

- The algorithm produces a facet of maximal weight for every weight function w.
- $\mathscr{I}$ is a matroid independence system.

†Recall that "maximal" means "not contained in any other element of $\mathscr{I}$", which is a logically weaker condition than "of largest possible cardinality".