Math 824, Fall 2010
Problem Set #1
Due date: Friday 9/3/10 in class

Problem #1 A *directed acyclic graph* or DAG, is a pair $G = (V, E)$, where V is a finite set of *vertices*; E is a finite set of *edges*, each of which is an ordered pair of distinct vertices; and E contains no directed cycles, i.e., no subsets of the form

$$\{(v_1, v_2),\ (v_2, v_3),\ \ldots,\ (v_{n-1}, v_n),\ (v_n, v_1)\}$$

for any $v_1, \ldots, v_n \in V$.

(#1a) Let P be a poset with order relation $<$. Let $E = \{(v, w) \mid v, w \in P, v < w\}$. Prove that the pair (P, E) is a DAG.

(#1b) Let $G = (V, E)$ be a DAG. Define a relation $<$ on V by setting $v < w$ iff there is some directed path from v to w in G, i.e., iff E has a subset of the form $\{(v_1, v_2),\ (v_2, v_3),\ \ldots,\ (v_{n-1}, v_n)\}$ with $v = v_1$ and $w = v_n$. Prove that this relation makes V into a poset.

(This problem is purely a technical exercise, but it does show that posets and DAGs are essentially the same thing.)

Problem #2 Let n be a positive integer. Let D_n be the set of all divisors of n (including n itself), partially ordered by divisibility.

(#2a) Prove that D_n is a ranked poset, and describe the rank function.

(#2b) For which values of n is D_n (i) a chain; (ii) a Boolean algebra? For which values of n, m is it the case that $D_n \cong D_m$?

(#2c) Prove that D_n is a distributive lattice. Describe its meet and join operations and its join-irreducible elements.

(#2d) Prove that D_n is *self-dual*, i.e., there is a bijection $f : D_n \to D_n$ such that $f(x) \leq f(y)$ if and only if $x \geq y$.

Problem #3 Prove that if L is a lattice, then

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \in L$$

if and only if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \in L.$$

(A consequence is that L is distributive if and only if L^* is; that is, distributivity is a self-dual condition.)

Problem #4 **(#4a)** Describe the join-irreducible elements of Young's lattice Y.

(#4b) Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition, and let $\lambda = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_k$ be the unique minimal decomposition of λ into join-irreducibles. Explain how to find k from the Ferrers diagram of λ.

Problem #5 **(#5a)** Count the maximal chains in $L_n(q)$. (Recall that this is the lattice of vector subspaces of the finite field with q elements).

(#5b) Count the maximal chains in the interval $[\emptyset, \lambda] \subset Y$ if the Ferrers diagram of λ is a $2 \times n$ rectangle.

(#5c) Ditto if λ is a hook shape (i.e., $\lambda = (n + 1, 1, 1, \ldots, 1)$, with a total of m copies of 1).

Problem #6 Fill in the details in the proof of Birkhoff's theorem by showing the following facts.

(#6a) For a finite distributive lattice L, show that the map $\phi : L \to J(\mathrm{Irr}(L))$ given by

$$\phi(x) \;=\; \langle p \mid p \in \mathrm{Irr}(L),\ p \leq x \rangle$$

is indeed a lattice isomorphism.

(#6b) For a finite poset P, show that an order ideal in P is join-irreducible in $J(P)$ if and only if it is principal (i.e., generated by a single element).