Math 821, Spring 2014
Problem Set #3
Due date: Friday, February 28

Problem #1 Recall that for a space X and basepoint $p \in X$, we have defined $\pi_1(X, p)$ to be the set of homotopy classes of p,p-paths on X — or equivalently of continuous functions $S^1 \to X$. Recall also that S^0 consists of two points (let's call them a and b) with the discrete topology. Accordingly, we could define $\pi_0(X, p)$ to be the set of homotopy classes of continuous functions $f : S^0 \to X$ such that $f(a) = p$.

Describe the set $\pi_0(X, p)$ intrinsically in terms of X. Is there a natural way to endow it with a group structure?

Problem #2 (Hatcher, p.38, #2) Show that the change-of-basepoint homomorphism β_h (see p.28) depends only on the homotopy class of the path h.

Problem #3 (Hatcher, p.38, #7) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on *one* of the boundary circles, but not by any homotopy f_t that is stationary on *both* boundary circles.

Problem #4 [Hatcher p.38 #8] Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$? Why or why not?

Problem #5 [Hatcher p.39 #9] Use the 2-dimensional case of the Borsuk-Ulam theorem (Hatcher, Thm. 1.10, p.32) to prove the "Ham and Cheese Sandwich Theorem: if A_1, A_2, A_3 are compact (hence measurable) sets in $\mathbb{R}^3$, then there is a plane in $\mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Problem #6 [Hatcher p.39 #12] Fix $p \in S^1$. Show that every homomorphism $\pi_1(S^1, p) \to \pi(S^1, p)$ can be realized as the induced homomorphism ϕ_* for some $\phi : S^1 \to S^1$.

Problem #7 [Hatcher, p.52, #1] Recall that the **center** of a group G is defined as $Z(G) = \{g \in G : gh = hg \ \forall h \in G\}$.

(#7a) Show that the free product $G * H$ of nontrivial groups G and H has trivial center.

(#7b) Show that the only elements of $G * H$ of finite order are the conjugates of finite-order elements in $G \cup H$.