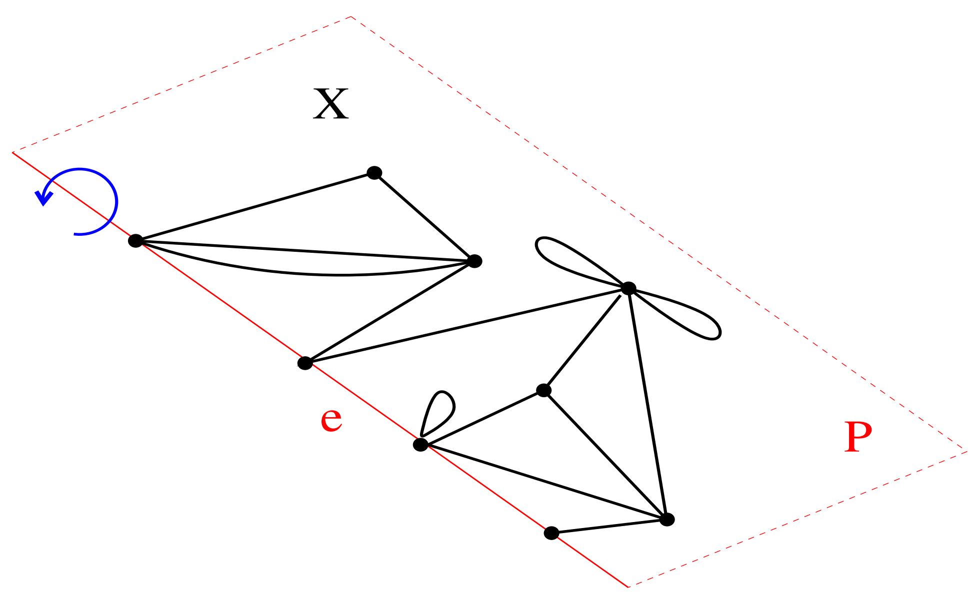
Math 821, Spring 2014
Problem Set #2
Due date: Friday, February 14

Problem #1 (Hatcher, p.19, #12) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y that is homotopy-equivalent to X.

Problem #2 Let p and q be two distinct points on S^2, and let X be the space obtained by gluing them together. Show that X is homotopy-equivalent to a wedge of spheres, and determine its exact homotopy type.

Problem #3 For $n \geq 1$, let T_n denote the n-holed torus. Construct a cell complex structure on T_n. (You can do this by a picture.)

Problem #4 (Hatcher, p.20, #22) Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge e of P in a subset of its vertices. (See example below.) Describe the homotopy type of the "surface of revolution" obtained by rotating X about e.



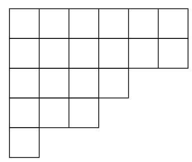
Problem #5 Let $0 \leq k \leq n$. Recall from class that the *Grassmannian* $G(k,n)$ is defined as the space of k-dimensional subspaces $V \subset \mathbb{R}^n$, so that in particular, $G(1,\mathbb{R}^n) = \mathbb{R}P^{n-1}$. (Fact: Everything in this problem works the same way if you change $\mathbb{R}$ to $\mathbb{C}$, except that the dimensions of all the cells get doubled.)

(#5a) Work out an explicit cell decomposition for $G(2,4)$ as a finite CW-complex. That is, describe how to decompose the set $G(2,4)$ into pieces, each of which is isomorphic to a $\mathbb{R}$-vector space. If you do this correctly (hint: row-reduced echelon form), then the isomorphisms should be straightforward from the construction.

(#5b) Describe the attaching poset of $G(2,4)$. (This is the partially ordered set whose elements are the cells e_α, and whose order relation is given by $e_\alpha \geq e_\beta$ if $\overline{e_\alpha} \supseteq e_\beta$).

(#5c) Describe the attaching poset of $G(2,5)$.

(#5d) A *Ferrers diagram* is a collection of square boxes that are top- and right-justified: for instance,



Write out the poset $P(2,3)$ of all Ferrers diagrams with at most two rows and at most three columns, ordered by containment (as sets of squares). Compare it to your previous answer.

Problem #6 [**Extra credit**] [Hatcher p.19, #20] Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure on p.19 of Hatcher, is homotopy equivalent to $X^1 \vee X^1 \vee S^2$. (Note: This is extra credit because currently I don't know how to do it. I'm hoping someone can show me.)