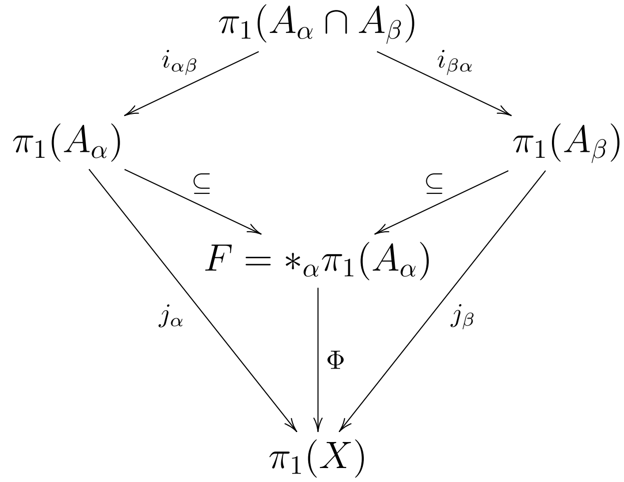
7. Monday 2/24: Van Kampen's Theorem — The Proof

Recall the statement of Van Kampen's Theorem.

Let $p \in X$, and let $\{A_\alpha : \alpha \in \mathscr{A}\}$ be a cover of X by path-connected open sets such that $p \in A_\alpha$ for every α. We have a commutative diagram of groups, which looks in part like this (where the i's and j's are the group homomorphisms induced by inclusions of spaces).

(7.1)



$\pi_1(-) = \pi_1(-, p)$

Van Kampen's Theorem:

(1) If every pairwise intersection $A_\alpha \cap A_\beta$ is path-connected, then the map Φ is surjective.

(2) If in addition every triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then

$$\ker \Phi = N := \langle\!\langle i_{\alpha\beta}[f] * i_{\beta\alpha}[\bar{f}] \ : \ \alpha, \beta \in \mathscr{A} \rangle\!\rangle$$

and so

$$\pi_1(X) = F/N.$$

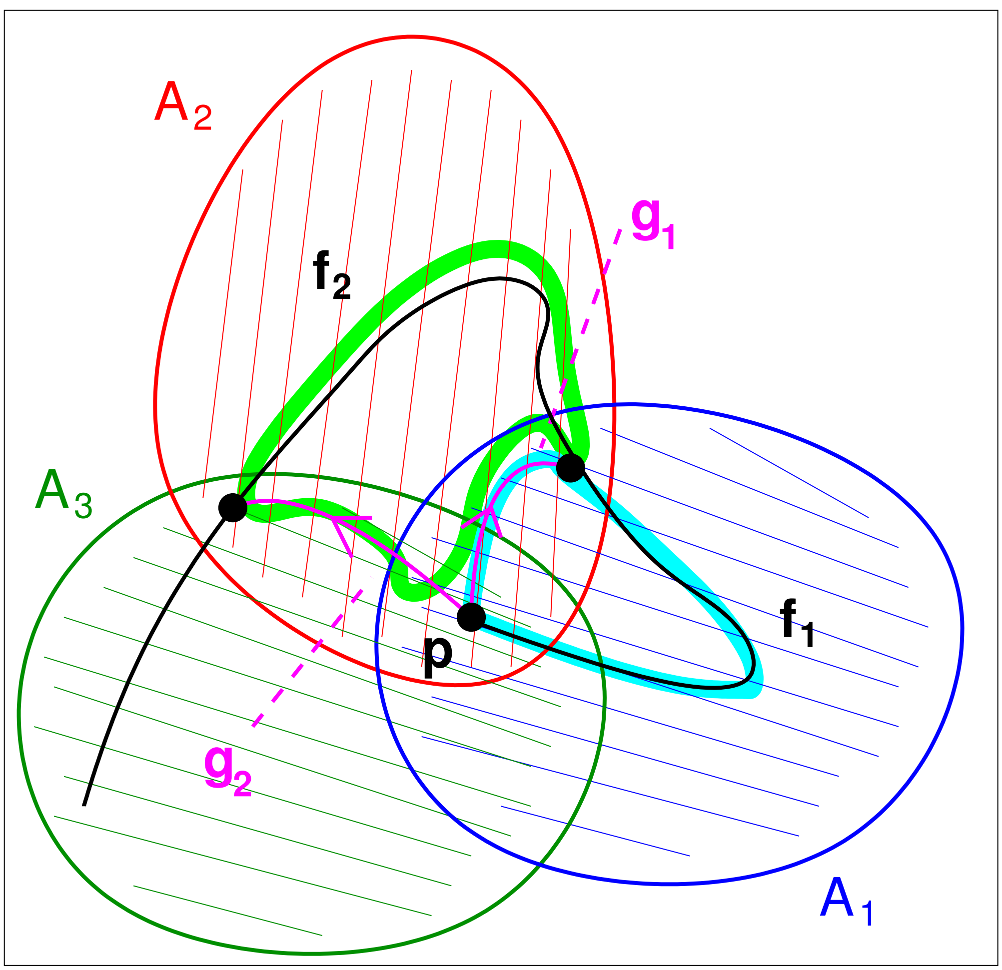
Proof of (1). Let $f : I \to X$ be a loop based at p. Every $s \in I$ has a neighborhood mapped by f into some U_α. By compactness of I, there exist numbers $0 = s_0 < s_1 < \cdots < s_m = 1$ and indices $\alpha_1, \ldots, \alpha_m$ such that

$$f([s_{i-1}, s_i]) \subseteq A_{\alpha_i} \quad \forall i \in [m].$$

Let $f_i = f|_{[s_{i-1},s_i]}$, so that $f = f_1 \cdot f_2 \cdots f_m$. For each $i \in [m]$, the set $A_i \cap A_{i+1}$ is path-connected, hence contains a path g_i from p to $f(s_i)$. Therefore

$$\begin{aligned} f &= f_1 \cdot f_2 \cdots f_m \\ &= (f_1 \cdot \overline{g_1}) \cdot (g_1 \cdot f_2 \cdot \overline{g_2}) \cdots (g_{m-2} \cdot f_{m-1} \cdot \overline{g_{m-1}}) \cdot (g_m \cdot f_m) \\ &\in \pi_1(A_1, p) * \pi_1(A_2, p) * \cdots * \pi_1(A_m, p) \\ &\in \operatorname{im} \Phi. \end{aligned}$$

□

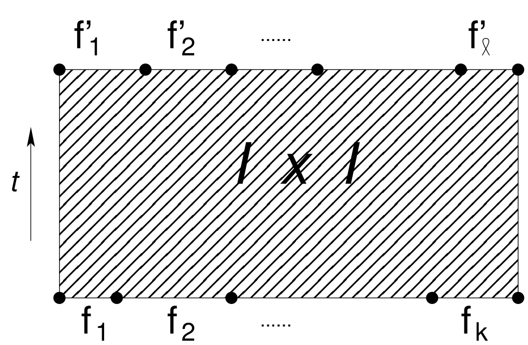


Proof of (2). Let $[f] \in \pi_1(X)$. Say that a **factorization** of $[f]$ is an expression $[f_1] * [f_2] * \cdots * [f_n]$ that maps to $[f]$ via Φ. Here I am using $*$ to denote concatenation of letters to make a word in $*_\alpha \pi_1(A_\alpha)$. That is, each $[f_i]$ belongs to some $\pi_1(A_\alpha)$, and $f \simeq f_1 \cdot f_2 \cdots f_n$.

We want to show that any two factorizations of $[f]$ are related by operations of the following forms:

- "Type A": If $f_i : I \to A_\alpha \cap A_\beta$, then we can regard the letter $[f_i]$ as coming either from $\pi_1(A_\alpha)$ or from $\pi_1(A_\beta)$. This amounts to inserting an element of N into f, namely
$$i_{\alpha\beta}[f_i] * i_{\beta\alpha}[\overline{f_i}].$$
- "Type B": If two consecutive letters in the factorization come from the same A_α, we can multiply them. This, of course, doesn't change the element of F we're talking about.

So, suppose we have two factorizations
$$[f] \;=\; \Phi\left([f_1] * \cdots * [f_k]\right) \;=\; \Phi\left([f'_1] * \cdots * [f'_\ell]\right).$$
In particular, there is a path-homotopy of p-loops $H : I \times I \to X$, $h_t(s) = H(s,t)$, such that
$$h_0 = f_1 \cdots f_k \quad \text{and} \quad h_1 = f'_1 \cdots f'_\ell.$$
Schematically, here's what this looks like:

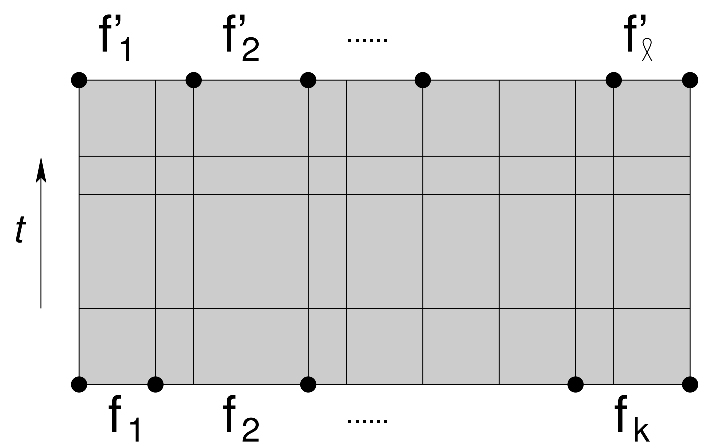


The dots on the top and bottom lines are the breakpoints between successive f_i's or f'_i's.

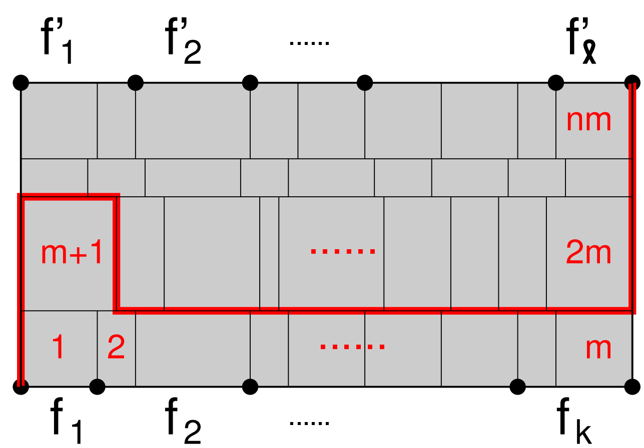
Now, we do something clever. Partition $I \times I$ into a finite grid of finitely many little rectangles R_i such that

$$\forall R_i : \ \exists i \in \mathscr{A} : \ H(R_i) \subset A_i. \tag{7.2}$$

(By continuity of H, we can put such a rectangle around each point in $I \times I$, then choose a finite subcover, then subdivide if necessary.) Subdivide more by adding vertical lines at all the breakpoints, and at least two horizontal lines.



Now, we do something *exceedingly* clever. For all of the vertical lines not in the first or last row, give them a little nudge to one side so they don't match up. We can do this while still retaining the condition (7.2). Number the rectangles $R_1, \ldots, R_{mn}$ as shown, where m is the number of columns and n is the number of rows.



Let γ_k be the path from $(0,0)$ to $(1,1)$ along the cell walls that separates rectangles $R_1, \dots, R_k$ from $R_{k+1}, \dots, R_{mn}$. (For example, the thick red path shown in the figure above is R_{m+1}.) Thus $H \circ \gamma_k$ is a closed path in X with basepoint p, and all the paths $H \circ \gamma_k$ are path-homotopic.

Each γ_k can be written as

$$\gamma_k = e_1 \cdot e_2 \cdots e_N$$

where each e_i is the path in X given by part of a side of one rectangle, say from v_{i-1} to v_i.

For each v_i, choose some path g_i in X from p to $F(v_i)$. Each v_i belongs to at most three rectangles, so we can require g_i to stay in the intersection of the corresponding three A's. (Wasn't that clever of us?)

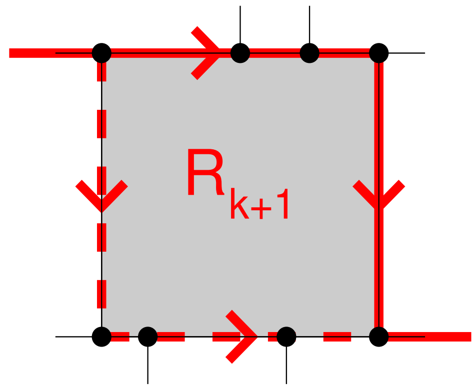
Then each γ_k can be factored as

$$\begin{aligned}
\gamma_k &= e_1 \cdots e_N \\
&= \Phi(e_1 * \cdots * e_N) \\
&= \Phi\Big([e_1 \cdot g_1] * [\overline{g_1} \cdot e_2 \cdot g_2] * \cdots * [\overline{g_{N-2}} \cdot e_{N-1} \cdot g_{N-1}] * [\overline{g_{N-1}} \cdot e_N] \Big)
\end{aligned}$$

Recall that $*$ means concatenation of letters in the free product F, while $\cdot$ means concatenation within one of its free factors.

To pass from the factorization for γ_k to that of γ_{k+1}, we have to trade the south and west sides of R_{k+1} for the north and east sides. We can do this by

- regarding the letters in the south and west sides as now coming from $\pi_1(A_{k+1})$ instead of wherever they came from in the factorization of γ_k (this is a type-A move);
- using the group structure of $\pi_1(A_{k+1})$ to trade the letters in the south and west sides for the north and east ones (this is a type-B move).



Now let's look at the path γ_0, which consists of the bottom and right edges of $I \times I$. The right edge is a stationary path, so forget about it. For each vertex v_i on the bottom edge of $I \times I$, we have so far only required g_i to lie in **two** of the A's. Let's also require it to lie in the same one whose fundamental group contains the letter f_i (which came from the factorization of f given in advance). That says that

For example, if $f_3 = e_1 \cdot e_2 \cdot e_3$, then the factorization begins

$$[e_1 \cdot g_1] * [\overline{g_1} \cdot e_2 \cdot g_2] * [\overline{g_2} \cdot e_3 \cdot g_3] * \cdots$$

where g_1 is a path in $A_1 \cap A_2$ and g_2 is a path in $A_2 \cap A_3$. But in fact we can require g_1 and g_2 to be paths in $A_1 \cap A_2 \cap A_\alpha$ and $A_2 \cap A_3 \cap A_\alpha$, where $\pi_1(A_\alpha)$ is the group containing the letter f_3. We also may as well assume that g_3 is the stationary path. So the partial factorization shown above can be replaced (with type-A moves) with one in $\pi_1(A_\alpha)$, and then simplified to the single letter $[e_1 \cdot e_2 \cdot e_3] = [f_1] \in \pi_1(A_\alpha)$.

More generally, if v_i is a breakpoint then we take g_i to be the constant path, and if v_i is not a breakpoint then we require g_i to lie in A_α for whichever $\pi_1(A_\alpha)$ contains the letter f_j to which the edges at v_i contribute. Then parenthesizing the factorization of f at the breakpoints shows that it is equivalent to $[f_1] * \cdots * [f_k]$.

Playing the same game at the top of the square shows that the factorization of γ_{nm} is equivalent to $[f'_1] * \cdots * [f'_\ell]$. □

□