Friday 4/25

Symmetric Functions

We continue our catalog of important symmetric functions. We have already seen (1) the *monomial symmetric functions*

$$m_\lambda = \sum_{\{a_1,\dots,a_\ell\}\subset\mathbb{P}} x_{a_1}^{\lambda_1} x_{a_2}^{\lambda_2} \cdots x_{a_\ell}^{\lambda_\ell};$$

(2) the *elementary symmetric functions*

$$e_k = \sum_{0<i_1<i_2<\dots<i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell};$$

and (3) the *complete homogeneous symmetric functions*

$$h_k = \sum_{0<i_1\le i_2\le\dots\le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

4. Power sums. These are defined by

$$p_k = x_1^k + x_2^k + \cdots = m_k,$$
$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

For example, in degree 2,

$$p_2 = m_2,$$
$$p_{11} = (x_1 + x_2 + \cdots)^2 = m_2 + 2m_{11}.$$

While $\{p_2, p_{11}\}$ is a $\mathbb{Q}$-vector space basis for $\Lambda_\mathbb{Q}$, it is not a $\mathbb{Z}$-module basis for $\Lambda_\mathbb{Z}$. To put this in a more elementary way, not every symmetric function with integer coefficients can be expressed as an integer combination of the power-sums; for example, $m_{11} = (p_{11} - p_2)/2$.

5. Schur functions. The definition of these power series is very different from the preceding ones, and it looks quite weird at first. However, the Schur functions turn out to be essential in the study of symmetric functions.

Definition 1. A **column-strict tableau** T of shape λ, or **λ-CST** for short, is a labeling of the boxes of a Ferrers diagram with integers (not necessarily distinct) that is

- weakly increasing across every row; and
- strictly increasing down every column.

The partition λ is called the **shape** of T, and the set of all column-strict tableaux of shape λ is denoted $CST(\lambda)$. The **content** of a CST is the sequence $\alpha = (\alpha_1, \alpha_2, \dots)$, where α_i is the number of boxes labelled i, and the **weight** of T is the monomial $x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

For example, here are two CST's, and one tableau that is not an CST, of shape $\lambda = (3, 2)$:

1	1	3
2	3	

$x_1^2\, x_2\, x_3^2$

1	1	1
4	8	

$x_1^3 x_4 x_8$

1	2	3
1	4	

Not a SST

Definition 2. The **Schur function** corresponding to a partition λ is

$$s_\lambda \;=\; \sum_{T \in CST(\lambda)} x^T.$$

It is far from obvious that s_λ is symmetric, but in fact it is. We will prove this shortly.

Example 1. Suppose that $\lambda = (n)$ is the partition with one part, so that the corresponding Ferrers diagram has a single row. Each multiset of n positive integers (with repeats allowed) corresponds to exactly one CST, in which the numbers occur left to right in increasing order. Therefore

$$s_{(n)} = h_n = \sum_{\lambda \vdash n} m_\lambda. \tag{1}$$

At the other extreme, suppose that $\lambda = (1, 1, \ldots, 1)$ is the partition with n singleton parts, so that the corresponding Ferrers diagram has a single column. To construct a CST of this shape, we need n distinct labels, which can be arbitrary. Therefore

$$s_{(1,1,\ldots,1)} = e_n = m_{(1,1,\ldots,1)}. \tag{2}$$

Let $\lambda = (2, 1)$. We will express s_λ as a sum of monomial symmetric functions. No tableau in $CST(\lambda)$ can have three equal entries, so the coefficient of m_3 is zero.

For weight $x_a x_b x_c$ with $a < b < c$, there are two possibilities, shown below.

a	b
c	

a	c
b	

Therefore, the coefficient of m_{111} is 1.

Finally, for every $a \neq b \in \mathbb{N}$, there is one tableau of shape λ and weight $x_a^2 x_b$ — either the one on the left if $a < b$, or the one on the right if $a > b$.

a	b
b	

b	b
a	

Therefore, $s_{(2,1)} \;=\; 2m_{111} + m_{21}$.

Proposition 1. *s_λ is a symmetric function for all λ.*

Proof. First, observe that the number

$$c(\lambda, \alpha) = |\{T \in CST(\lambda) \mid x^T = x^\alpha\}| \tag{3}$$

depends only on the ordered sequence of nonzero exponents[*] in α. For instance, for any $\lambda \vdash 8$, there are the same number of λ-CST's with weights

$$x_1^1 x_2^2 x_3^4 x_9^1 \quad \text{and} \quad x_1^1 x_2^2 x_7^4 x_9^1$$

because there is an obvious bijection between them given by changing all 3's to 7's or vice versa.

To complete the proof that s_λ is symmetric, it suffices to show that swapping the powers of adjacent variables does not change $c(\lambda, \alpha)$. That will imply that s_λ is invariant under every adjacent transposition $(k\ k+1)$, and these transpositions generate the group $\mathfrak{S}_\infty$.

[*] This is precisely the statement that s_λ is a *quasisymmetric* function.

We will prove this by a bijection, which is easiest to show by example. Let $\lambda = (9, 7, 4, 3, 2)$. We would like to show that there are the same number of λ-CST's with weights

$$x_1^3 x_2^2 x_3^3 x_4^3 \boldsymbol{x_5^4 x_6^7} x_7^3 \quad \text{and} \quad x_1^3 x_2^2 x_3^3 x_4^3 \boldsymbol{x_5^7 x_6^4} x_7^3.$$

Let T be the following λ-CST:

1	1	1	2	3	5	6	6	6
2	3	4	5	6	7	7		
3	4	5	6					
4	6	6						
5	7							

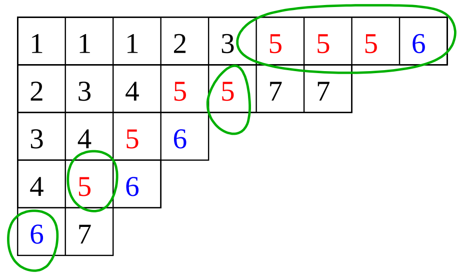
Observe that the occurrences of 5 and of 6 each form "snakes" from southwest to northeast.

1	1	1	2	3	5	6	6	6
2	3	4	5	6	7	7		
3	4	5	6					
4	6	6						
5	7							

To construct a new tableau in which the numbers of 5's and of 6's are switched, we ignore all the columns containing both a 5 and a 6, and then group together all the other strings of 5's and 6's in the same row.

1	1	1	2	3	5	6	6	6
2	3	4	5	6	7	7		
3	4	5	6					
4	6	6						
5	7							

Then, we swap the numbers of 5's and 6's in each of those contiguous blocks.



This construction allows us to swap the exponents on x_k and x_{k+1} for any k, concluding the proof. $\square$

Theorem 2. *For each $n \geq 1$, the sets*

$$\{m_\lambda \mid \lambda \vdash n\}, \quad \{e_\lambda \mid \lambda \vdash n\}, \quad \{h_\lambda \mid \lambda \vdash n\}, \quad \text{and} \quad \{s_\lambda \mid \lambda \vdash n\}$$

are all $\mathbb{Z}$-bases for Λ, i.e., bases for $\Lambda_{\mathbb{Z},n}$ as a free $\mathbb{Z}$-module, and

$$\{p_\lambda \mid \lambda \vdash n\}$$

is a $\mathbb{Q}$-basis for Λ, i.e., a basis for $\Lambda_{\mathbb{Q},n}$ as a vector space. Moreover,

$$\{e_1, e_2, \ldots\} \quad \text{and} \quad \{h_1, h_2, \ldots\}$$

generate Λ as a polynomial algebra over R.

Sketch of proof: It is more or less obvious that the m_λ are a $\mathbb{Z}$-basis.

To show that the Schur functions are a $\mathbb{Z}$-basis, we show that they can be obtained from m_λ by a *unitriangular* change of basis. Specifically, we write each Schur function as an integer linear combination of monomial symmetric functions as

$$s_\lambda = \sum_{\lambda \vdash n} K_{\lambda\mu} m_\mu$$

and then show that the matrix $[K_{\lambda\mu}]$ is triangular, with 1's on the main diagonal; therefore, it is invertible and its inverse has integer entries. Note that by the definition of Schur functions, the coefficient $K_{\lambda\mu}$ is the number of column-strict tableaux with shape λ and content μ; these are the so-called **Kostka numbers**.

Of course, to do this we have to specify an ordering on the partitions. Rather than the lexicographic total order we have worked with before, it turns out to be convenient to work with a *partial* order, as follows.

Definition 3. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be partitions of n. We say that λ **dominates** μ, written $\lambda \trianglerighteq \mu$, if $\ell \leq m$ and

$$\begin{gathered} \lambda_1 \geq \mu_1, \\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \\ \cdots \\ \lambda_1 + \cdots + \lambda_\ell \geq \mu_1 + \cdots + \mu_\ell. \end{gathered}$$

Proposition 3. $K_{\lambda\lambda} = 1$ *for all λ. Moreover, $K_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$.*

The proof of this fact is a homework problem. As a corollary, the matrix of Kostka numbers is unitriangular for any total order (such as the lexicographic order) which refines dominance. A similar result holds for the elementary symmetric functions. If we write

$$e_\lambda = \sum_\mu B_{\lambda\mu} m_\mu$$

then the coefficients $B_{\lambda\mu}$ have a nice combinatorial interpretation, and it turns out that

$$B_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu = \lambda', \\ 0 & \text{if } \mu \not\trianglelefteq \lambda' \end{cases}$$

where λ' denotes the conjugate (or transpose) of λ.

The proof that the p_λ form a $\mathbb{Q}$-basis is analogous, although in this case the change of basis has non-1's on the diagonal and so is not invertible over $\mathbb{Z}$ (but it is invertible over $\mathbb{Q}$).

The h_λ's are different. They have lots and lots of terms, so the coefficients of the transition matrix are all nonzero and we can't use triangularity to prove that they are a basis. However, we can do something else clever.