Wednesday 4/23

Frobenius Reciprocity

Let $H \subset G$ be finite groups, and let ψ, χ be characters of G and H respectively. The restricted character of ψ on H is

$$\operatorname{Res}_H^G \psi(h) = \psi(h) \tag{1}$$

and the induced character of χ on G is

$$\operatorname{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{\substack{k \in G \\ k^{-1}gk \in H}} \chi_\rho(k^{-1}gk). \tag{2}$$

Theorem 1 (Frobenius Reciprocity)**.** $\left\langle \operatorname{Ind}_H^G \chi,\, \psi \right\rangle_G = \left\langle \chi,\, \operatorname{Res}_H^G \psi \right\rangle_H.$

Proof.

$$\begin{aligned}
\left\langle \operatorname{Ind}_H^G \chi,\, \psi \right\rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\operatorname{Ind}_H^G \chi(g)} \psi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{1}{|H|} \sum_{k \in G:\ k^{-1}gk \in H} \overline{\chi(k^{-1}gk)} \psi(g) && \text{(by (2))} \\
&= \frac{1}{|G||H|} \sum_{h \in H} \sum_{k \in G} \sum_{\substack{g \in G \\ k^{-1}gk = h}} \overline{\chi(h)} \psi(k^{-1}gk) \\
&= \frac{1}{|G||H|} \sum_{h \in H} \sum_{k \in G} \overline{\chi(h)} \psi(h) && \text{(i.e., } g = khk^{-1}\text{)} \\
&= \frac{1}{|H|} \sum_{h \in H} \overline{\chi(h)} \psi(h) \\
&= \left\langle \chi,\, \operatorname{Res}_H^G \psi \right\rangle_H.
\end{aligned}$$

□

See Monday's notes for an application (and there will be more later).

Symmetric Functions

Definition 1. Let R be a commutative ring (typically $\mathbb{Q}$ or $\mathbb{Z}$). A *symmetric function* is a polynomial in $R[x_1, \ldots, x_n]$ that is invariant under permuting the variables.

For example, if $n = 3$, then up to scalar multiplication, the only symmetric function of degree 1 in x_1, x_2, x_3 is $x_1 + x_2 + x_3$.

In degree 2, here are two:

$$x_1^2 + x_2^2 + x_3^2, \qquad x_1x_2 + x_1x_3 + x_2x_3.$$

Every other symmetric function that is homogeneous of degree 2 is a R-linear combination of these two, because the coefficients of x_1^2 and x_1x_2 determine the coefficients of all other monomials. Note that the set of all degree-2 symmetric functions forms a vector space.

In degree 3, the following three polynomials form a basis for the space of symmetric functions:

$$x_1^3 + x_2^3 + x_3^3,$$
$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2,$$
$$x_1 x_2 x_3.$$

Each member of this basis is a sum of the monomials in a single orbit under the action of $\mathfrak{S}_3$. Accordingly, we call them **monomial symmetric functions**, and index each by the partition whose parts are the exponents of one of its monomials. That is,

$$\begin{aligned} m_3(x_1, x_2, x_3) &= x_1^3 + x_2^3 + x_3^3, \\ m_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2, \\ m_{111}(x_1, x_2, x_3) &= x_1 x_2 x_3. \end{aligned}$$

In general, for $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define

$$m_\lambda(x_1, \ldots, x_n) = \sum_{\{a_1,\ldots,a_\ell\} \subset [n]} x_{a_1}^{\lambda_1} x_{a_2}^{\lambda_2} \cdots x_{a_\ell}^{\lambda_\ell}.$$

But unfortunately, this is zero if $\ell > n$. So we need more variables! In fact, we will in general work with an *infinite** set of variables $\{x_1, x_2, \ldots\}$.

Definition 2. Let $\lambda \vdash n$. The **monomial symmetric function** m_λ is the power series

$$m_\lambda = \sum_{\{a_1,\ldots,a_\ell\} \subset \mathbb{P}} x_{a_1}^{\lambda_1} x_{a_2}^{\lambda_2} \cdots x_{a_\ell}^{\lambda_\ell}.$$

That is, m_λ is the sum of all monomials whose exponents are the parts of λ. Another way to write this is

$$m_\lambda = \sum_{\substack{\text{rearrangements} \\ \alpha \text{ of } \lambda}} x^\alpha$$

where x_α is shorthand for $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. Here we are regarding λ as a countably infinite sequence in which all but finitely many terms are 0.

We then define

$$\begin{aligned} \Lambda_d = \Lambda_{R,d} &= \{\text{degree-}d \text{ symmetric functions with coeff'ts in } R\}, \\ \Lambda = \Lambda_R &= \bigoplus_{d \geq 0} \Lambda_d. \end{aligned}$$

Each Λ_d is a finite-dimensional vector space, with basis $\{m_\lambda \mid \lambda \vdash d\}$. $\dim_{\mathbb{C}} \Lambda_d = p(d)$ (the number of partitions of d), and the dimension does not change even if we zero out all but d variables, so for many purposes it is permissible (and less intimidating) to regard Λ_d as the space of degree-d symmetric functions in d variables.

Moreover, Λ is a graded ring. In fact, let $\mathfrak{S}_\infty$ be the group whose members are the permutations of $\{x_1, x_2, \ldots\}$ with only finitely many non-fixed points; that is,

$$\mathfrak{S}_\infty = \bigcup_{n=1}^{\infty} \mathfrak{S}_n.$$

Then

$$\Lambda = R[[x_1, x_2, \ldots,]]^{\mathfrak{S}_\infty}.$$

*This (understandably) bothers some people. In practice, we rarely have to worry about more than finitely many variables when carrying out calculations.

Where is all this going?

The punchline is that we are going to construct an isomorphism

$$\Lambda_{\mathbb{Q}} \xrightarrow{F} \bigoplus_{n\geq 0} C\ell_{\mathbb{Q}}(\mathfrak{S}_n)$$

called the *Frobenius characteristic*. Thus will allow us to translate symmetric function identities into statements about representations and characters of $\mathfrak{S}_n$, and vice versa.

Important Families of Symmetric Functions

Throughout this section, let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell) \vdash n$.

1. Monomial symmetric functions. These we have just seen.

2. Elementary symmetric functions. For $k \in \mathbb{N}$ we define

$$e_k \quad = \quad \sum_{\substack{S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} x_s \quad = \quad \sum_{0<i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \quad = \quad m_{11\cdots 1}$$

where there are k $1's$ in the last expression. (In particular $e_0 = 1$.) We then define

$$e_\lambda = e_{\lambda_1}\cdots e_{\lambda_\ell}.$$

For example,

$$\begin{aligned} e_{11} &= (x_1+x_2+x_3+\cdots)^2 \\ &= (x_1^2+x_2^2+\cdots) + 2(x_1x_2+x_1x_3+x_2x_3+x_1x_4+\cdots) \\ &= m_2 + 2m_{11}, \\ e_{21} &= (x_1+x_2+x_3+\cdots)(x_1x_2+x_1x_3+x_2x_3+x_1x_4+\cdots) \\ &= m_{21} + 3m_{111}, \\ e_{111} &= (x_1+x_2+x_3+\cdots)^3 \\ &= m_3 + 3m_{21} + 6m_{111}, \end{aligned}$$

et cetera.

Observe that

(3) $$E(t) := \prod_{i\geq 1}(1+tx_i) = \sum_{k\geq 0} t^k e_k.$$

3. (Complete) homogeneous symmetric functions. For $k \in \mathbb{N}$, we define h_k to be the sum of *all* monomials of degree k:

$$h_k \quad = \quad \sum_{\substack{\text{multisets } S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} x_s \quad = \quad \sum_{0<i_1\leq i_2\leq\cdots\leq i_k} x_{i_1}x_{i_2}\cdots x_{i_k} \quad = \quad \sum_{\lambda\vdash k} m_\lambda.$$

We then define

$$h_\lambda = h_{\lambda_1}\cdots h_{\lambda_\ell}.$$

For example, $h_{11} = e_{11}$ and

$$h_{21} \;=\; h_1 h_2 \;=\; e_1(m_{11} + m_2) \;=\; e_1(e_{11} - e_2) \;=\; e_{111} - e_{21} = m_3 + 2m_{21} + 3m_{111}.$$

The analogue of (4) for the homogeneous symmetric functions is

$$H(t) := \prod_{i \geq 1} \frac{1}{1 - tx_i} = \sum_{k \geq 0} t^k h_k. \tag{4}$$

In many situations, the elementary and homogeneous symmetric functions behave dually.

As we will see, the sets $\{e_\lambda \mid \lambda \vdash d\}$ and $\{h_\lambda \mid \lambda \vdash d\}$ are $\mathbb{Z}$-module bases for Λ_d.