Friday 4/18

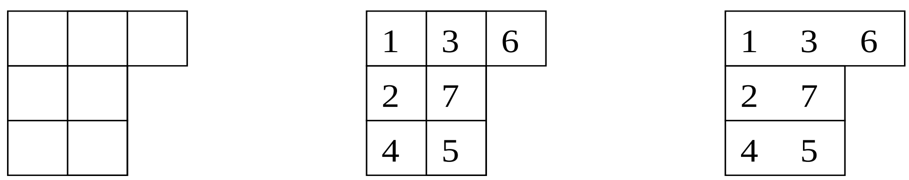
Characters of the Symmetric Group

We worked out the irreducible characters of $\mathfrak{S}_4$ *ad hoc*. We'd like to have a way of calculating them in general.

Recall that a *partition* of n is a sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of weakly decreasing positive integers whose sum is n. We write $\lambda \vdash n$ to indicate that λ is a partition of n. The number of partitions of n is denoted $p(n)$.

For $\lambda \vdash n$, let C_λ be the conjugacy class in $\mathfrak{S}_n$ consisting of all permutations with cycle shape λ. Since the conjugacy classes are in bijection with the partitions of n, it makes sense to look for a set of representations indexed by partitions.

Definition 1. Let $\mu = (\mu_1, \dots, \mu_m) \vdash n$. The **Ferrers diagram** of shape μ is the top- and left-justified array of boxes with μ_i boxes in the i^{th} row. A **Young tableau of shape** μ is a Ferrers diagram with the numbers $1, 2, \dots, n$ placed in the boxes, one number to a box. Two tableaux T, T' of shape μ are *row-equivalent*, written $T \sim T'$, if the numbers in each row of T are the same as the numbers in the corresponding row of T'. A **tabloid** of shape μ is an equivalence class of tableaux under row-equivalence. A tabloid can be represented as a tableau without vertical lines separating numbers in the same row. We write $\mathrm{sh}(T) = \mu$ to indicate that a tableau or tabloid T is of shape μ.



Ferrers diagram   Young tableau   Young tabloid

A Young tabloid can be regarded as a set partition $(T_1, \dots, T_m)$ of $[n]$, in which $|T_i| = \mu_i$. The order of the blocks T_i matters, but not the order of digits within each block. Thus the number of tabloids of shape μ is

$$\binom{n}{\mu} = \frac{n!}{\mu_1! \cdots \mu_m!}.$$

The symmetric group $\mathfrak{S}_n$ acts on tabloids by permuting the numbers. Accordingly, we have a permutation representation (ρ_μ, V^μ) of $\mathfrak{S}_n$ on the vector space V^μ of all $\mathbb{C}$-linear combinations of tabloids of shape μ.

Example 1. For $n = 3$, the characters of the representations ρ_μ are as follows.

(1)

	λ		
	111	21	3
$\mu = (3)$	1	1	1
$\mu = (2,1)$	3	1	0
$\mu = (1,1,1)$	6	0	0
$\lvert C_\lambda \rvert$	1	3	2

Many familiar representations of $\mathfrak{S}_n$ can be expressed in this form.

- There is a unique tabloid of shape $\mu = (n)$: $T = \boxed{1\ 2\ \cdots\ n}$. Every permutation fixes T, so

$$\rho_{(n)} \cong \rho_{\mathrm{triv}}.$$

- The tabloids of shape $\mu = (1, 1, \ldots, 1)$ are just the permutations of $[n]$. Therefore
$$\rho_{(1,1,\ldots,1)} \cong \rho_{\text{reg}}.$$
- A tabloid of shape $\mu = (n-1, 1)$ is determined by its singleton part. So the representation ρ_μ is isomorphic to the action of $\mathfrak{S}_n$ on this part by permutation; that is
$$\rho_{(n-1,1)} \cong \rho_{\text{def}}.$$

For $n = 3$, the table in (1) is triangular, which implies immediately that the characters ρ_μ are linearly independent. It's not hard to prove that this is the case for all n.

Definition 2. The *lexicographic order* on partitions $\lambda, \mu \vdash n$ is defined as follows: $\lambda > \mu$ if for some $k > 0$
$$\begin{aligned} \lambda_1 &= \mu_1, \\ \lambda_1 + \lambda_2 &= \mu_1 + \mu_2, \\ &\cdots \\ \lambda_1 + \cdots + \lambda_{k-1} &= \mu_1 + \cdots + \mu_{k-1}, \\ \lambda_1 + \cdots + \lambda_k &> \mu_1 + \cdots + \mu_k. \end{aligned}$$

Abbreviate χ_{ρ_μ} by χ_μ henceforth. Since the ρ_μ are permutation representations, we can calculate χ_μ by counting fixed points. That is,
$$\chi_\mu C_\lambda = \#\{\text{tabloids } T \mid \operatorname{sh}(T) = \mu,\ w(T) = T\}$$
for any $w \in C_\lambda$.

Proposition 1. *Let $\lambda, \mu \vdash n$. Then:*

(1) $\chi_\lambda(C_\lambda) \neq 0$.
(2) $\chi_\mu(C_\lambda) \neq 0$ *only if $\lambda \leq \mu$ in lexicographic order.*

Proof. To show that $\chi_\lambda(C_\lambda) \neq 0$, let $w \in C_\lambda$; we must find a tabloid T of shape λ fixed by w. Indeed, we can take T to be any tabloid whose blocks are the cycles of w. For example, if $w = (1\ 3\ 6)(2\ 7)(4\ 5) \in \mathfrak{S}_7$, then T can be either of the following two tabloids:

$$\begin{array}{|ccc|}\hline 1 & 3 & 6 \\ \hline 2 & 7 & \\ \hline 4 & 5 & \\ \hline \end{array} \qquad \begin{array}{|ccc|}\hline 1 & 3 & 6 \\ \hline 4 & 5 & \\ \hline 2 & 7 & \\ \hline \end{array}$$

On the other hand, w fixes a tabloid T of shape μ if and only if every cycle of w is contained in a row of P. In particular, the sum of any r parts of λ must be less than or equal to the sum of some r parts of μ, hence less than or equal to the sum of the first r parts of μ, which implies that $\lambda \leq \mu$ □

Corollary 2. *The characters $\{\chi_\mu \mid \mu \vdash n\}$ form a basis for $Cl(G)$.*

Proof. The number of these characters is $\dim C\ell(G)$. Moreover, Proposition 1 implies that the $p(n) \times p(n)$ matrix $X = [\chi_\mu(C_\lambda)]_{\mu,\lambda \vdash n}$ is triangular, hence nonsingular. □

We can transform the rows of the matrix X into a list of irreducible characters of $\mathfrak{S}_n$ by applying the Gram-Schmidt process (measuring orthogonality, of course, with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{S}_n}$). Indeed, the triangularity of X means that we will be able to label the irreducible characters of $\mathfrak{S}_n$ as
$$\{\tilde{\chi}_\nu \mid \nu \vdash n\}$$

so that

$$(2)\qquad \begin{aligned} \langle \tilde{\chi}_\nu,\, \chi_\nu \rangle_G &\neq 0, \\ \left\langle \tilde{\chi}_\nu,\, \chi_\mu \right\rangle_G &= 0 \qquad \text{if } \nu < \mu. \end{aligned}$$

Example 2. Recall the table of characters (1) of the representations ρ_μ for $n = 3$. We will use this to produce the table of irreducible characters. For brevity, let's omit the commas between the parts of partitions μ.

First, $\chi_{(3)} = [1, 1, 1] = \chi_{\text{triv}}$ is irreducible. We therefore call it $\tilde{\chi}_{(3)}$.

Second, for the character $\chi_{(21)}$, we observe that

$$\left\langle \chi_{(21)},\, \tilde{\chi}_{(3)} \right\rangle_G = 1.$$

Applying Gram-Schmidt, we construct a character orthonormal to $\tilde{\chi}_{(3)}$:

$$\tilde{\chi}_{(21)} \;=\; \chi_{(21)} - \tilde{\chi}_{(3)} \;=\; [2, 0, -1].$$

Notice that this character is irreducible.

Finally, for the character $\chi_{(111)}$, we have

$$\left\langle \chi_{(111)},\, \tilde{\chi}_{(3)} \right\rangle_G = 1,$$
$$\left\langle \chi_{(111)},\, \tilde{\chi}_{(21)} \right\rangle_G = 2.$$

Accordingly, we apply Gram-Schmidt to obtain the character

$$\tilde{\chi}_{(111)} \;=\; \chi_{(111)} - \tilde{\chi}_{(3)} - 2\tilde{\chi}_{(21)} \;=\; [1, -1, 1]$$

which is 1-dimensional, hence irreducible. In summary, the complete list of irreducible characters, labeled so as to satisfy (2), is as follows:

	λ			
	111	21	3	
$\tilde{\chi}_{(3)}$	1	1	1	$= \chi_{\text{triv}}$
$\tilde{\chi}_{(2,1)}$	2	0	-1	
$\tilde{\chi}_{(1,1,1)}$	1	-1	1	$= \chi_{\text{sign}}$

To summarize our calculation, we have shown that

$$\left[\chi_\mu\right]_{\mu\vdash 3} \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 6 & 0 & 0 \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{K} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \;=\; \left[K_{\lambda,\mu}\right]_{\lambda,\mu\vdash 3} \;\left[\tilde{\chi}_\lambda\right]_{\lambda\vdash 3}$$

that is,

$$\chi_\mu = \sum_\lambda K_{\lambda,\mu} \tilde{\chi}_\lambda.$$

The numbers $K_{\lambda,\mu}$ are called the **Kostka numbers**. We will eventually find a combinatorial interpretation for them, which will imply easily that the matrix K is unitriangular.