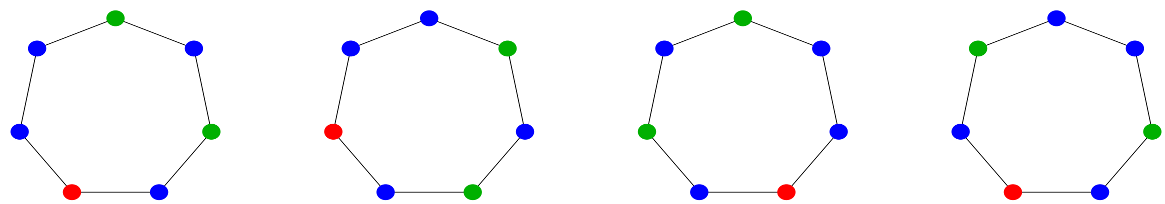
Friday 4/4

Group Actions and Polyá Theory

How many different necklaces can you make with four blue, two green, and one red bead?

It depends what "different" means. The second necklace can be obtained from the first by rotation, and the third by reflection, but the fourth one is honestly different from the first two.



If we just wanted to count the number of ways to permute four blue, two green, and one red beads, the answer would be the multinomial coefficient

$$\binom{7}{4,2,1} = \frac{7!}{4!\,2!\,1!} = 105.$$

However, what we are really trying to count is *orbits under a group action.*

Let G be a group and X a set. An **action** of G on X is a group homomorphism $\alpha : G \to \mathfrak{S}_X$, the group of permutations of X.

Equivalently, an action can also be regarded as a map $G \times X \to X$, sending (g, x) to gx, such that

- $1_G x = x$ for every $x \in X$ (where 1_G denotes the identity element of G);
- $g(hx) = (gh)x$ for every $g, h \in G$ and $x \in X$.

The *orbit* of $x \in X$ is the set

$$O_x = \{gx \mid g \in G\} \subset X$$

and its *stabilizer* is

$$S_x = \{g \in G \mid gx = x\} \subset G,$$

which is a subgroup of G.

To go back to the necklace problem, we now see that "same" really means "in the same orbit". In this case, X is the set of all 105 necklaces, and the group acting on them is the dihedral group D_7 (the group of symmetries of a regular heptagon). The number we are looking for is the number of orbits of D_7.

Lemma 1. *For every $x \in X$, we have $|O_x||S_x| = |G|$.*

Proof. The element gx depends only on which coset of S_x contains g, so $|O_x|$ is the number of cosets, which is $|G|/|S_x|$. $\square$

Proposition 2 (Burnside's Theorem)**.** *The number of orbits of the action of G on X equals the average number of fixed points:*

$$\frac{1}{|G|} \sum_{g \in G} \#\{x \in X \mid gx = x\}$$

Proof. For a sentence P, let $\chi(P) = 1$ if P is true, or 0 if P is false (the "Garsia chi function"). Then

$$\begin{aligned}
\text{Number of orbits} &= \sum_{x \in X} \frac{1}{|O_x|} = \frac{1}{|G|} \sum_{x \in X} |S_x| \\
&= \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} \chi(gx = x) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} \chi(gx = x) = \frac{1}{|G|} \sum_{g \in G} \#\{x \in X \mid gx = x\}.
\end{aligned}$$

□

Typically, it is easier to count fixed points than to count orbits directly.

Example 1. We can apply this technique to the necklace example above.

- The identity of D_7 has 105 fixed points.
- Each of the seven reflections in D_7 has three fixed points (the single bead lying on the reflection line must be red, and then the two green beads must be equally distant from it, one on each side).
- Each of the six nontrivial rotations has no fixed points.

Therefore, the number of orbits is

$$\frac{105 + 7 \cdot 3}{|D_7|} = \frac{126}{14} = 9,$$

which is much more pleasant than trying to count them directly.

Example 2. Suppose we wanted to find the number of orbits of 7-bead necklaces with 3 colors, without specifying how many times each color is to be used.

- The identity element of D_7 has $3^7 = 2187$ fixed points.
- Each reflection fixes one bead, which can have any color. There are then three pairs of beads flipped, and we can specify the color of each pair. Therefore, there are $3^4 = 81$ fixed points.
- Each rotation acts by a 7-cycle on the beads, so it has only three fixed points (all the beads must have the same color).

Therefore, the number of orbits is

$$\frac{2187 + 7 \cdot 81 + 6 \cdot 3}{14} = 198.$$

More generally, the number of inequivalent 7-bead necklaces with k colors allowed is

$$\frac{k^7 + 7k^4 + 6k}{14}. \tag{1}$$

As this example indicates, it is helpful to look at the cycle structure of the elements of G, or more precisely on their images $\alpha(g) \in \mathfrak{S}_X$.

Proposition 3. *Let X be a finite set, and let $\alpha : G \to \mathfrak{S}_X$ be a group action. Color the elements of X with k colors, so that G also acts on the colorings.*

1. For $g \in G$, the number of fixed points of the action of g is $k^\ell(g)$, where $\ell(g)$ is the number of cycles in the disjoint-cycle representation of $\alpha(g)$.

2. Therefore,

$$\#\textit{equivalence classes of colorings} = \frac{1}{|G|} \sum_{g \in G} k^{\ell(g)}. \tag{2}$$

Let's rephrase Example 2 in this notation. The identity has cycle-shape 1111111 (so $\ell = 7$); each of the six reflections has cycle-shape 2221 (so $\ell = 4$); and each of the seven rotations has cycle-shape 7 (so $\ell = 1$). Thus (1) is an example of the general formula (2).

Example 3. How many ways are there to k-color the vertices of a tetrahedron, up to moving the tetrahedron around in space?

Here X is the set of four vertices, and the group G acting on X is the alternating group on four elements. This is the subgroup of $\mathfrak{S}_4$ that contains the identity, of cycle-shape 1111; the eight permutations of cycle-shape 31; and the three permutations of cycle-shape 22. Therefore, the number of colorings is

$$\frac{k^4 + 11k^2}{12}.$$