Wednesday 3/26

Oriented Matroids

Last time:

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $\mathbb{R}^d$.

Let $\ell_1, \dots, \ell_n$ be affine linear forms such that $H_i = \{\vec{x} \in \mathbb{R}^d \mid \ell_i(\vec{x}) = 0\}$ for all i.

For $c = (c_1, \dots, c_n) \in \{+, -, 0\}^n$, let

$$F = \left\{ \vec{x} \in \mathbb{R}^d \mid \begin{array}{lll} \ell_i(\vec{x}) > 0 & \text{if} & c_i = + \\ \ell_i(\vec{x}) < 0 & \text{if} & c_i = - \\ \ell_i(\vec{x}) = 0 & \text{if} & c_i = 0 \end{array} \right\}$$

If $F \neq \emptyset$ then it is called a **face** of $\mathcal{A}$, and $c = c(F)$ is the corresponding **covector**.

$\mathscr{F}(\mathcal{A}) = \{\text{faces of } \mathcal{A}\}$

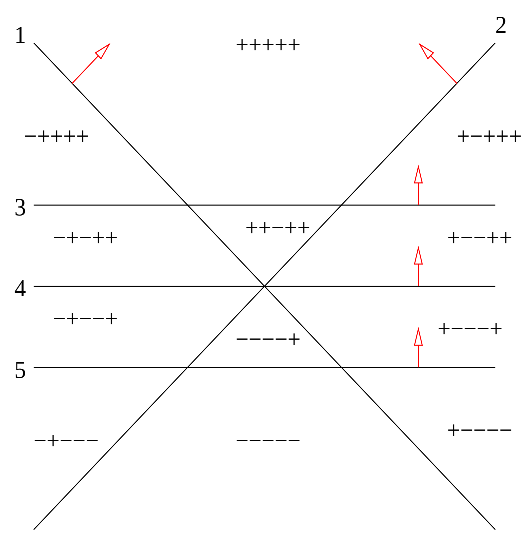
$\hat{\mathscr{F}}(\mathcal{A}) = \mathscr{F}(\mathcal{A}) \cup \{\hat{0}, \hat{1}\} =$ big face lattice of $\mathcal{A}$

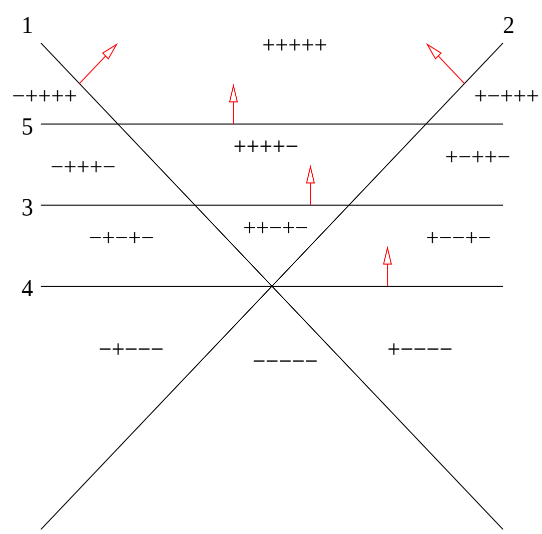
(ordered by $F \leq F'$ if $\bar{F} \subseteq \bar{F}'$).

Consider the linear forms ℓ_i that were used in representing each face by a covector. Specifying ℓ_i is equivalent to specifying a normal vector $\vec{v}_i$ to the hyperplane H_i (with $\ell_i(\vec{x}) = \vec{v}_i \cdot x$. As we know, the vectors $\vec{v}_i$ represent a matroid whose lattice of flats is precisely $L(\mathcal{A})$.

Scaling $\vec{v}_i$ (equivalently, ℓ_i) by a nonzero constant $\lambda \in \mathbb{R}$ has no effect on the matroid represented by the $\vec{v}_i$'s, but what does it do to the covectors? If $\lambda > 0$, then nothing happens, but if $\lambda < 0$, then we have to switch $+$ and $-$ signs in the i^{th} position of every covector. So, in order to figure out the covectors, we need not just the normal vectors $\vec{v}_i$, but an orientation for each one.

Example: Let's go back to the two arrangements considered at the start. Their regions are labeled by the following covectors:





Now, you should object that the oriented normal vectors are the same in each case. Yes, but this couldn't happen if the arrangements were *central*, because two vector subspaces of the same space cannot possibly be parallel. In fact, if $\mathcal{A}$ is a central arrangement, then the oriented normals determine $\mathscr{F}(\mathcal{A})$ uniquely.

Proposition: The covectors of $\mathcal{A}$ are preserved under the operation of *negation* (changing all $+$'s to $-$'s and vice versa) if and only if $\mathcal{A}$ is central. In fact, the maximal covectors that can be negated are exactly those that correspond to bounded regions.

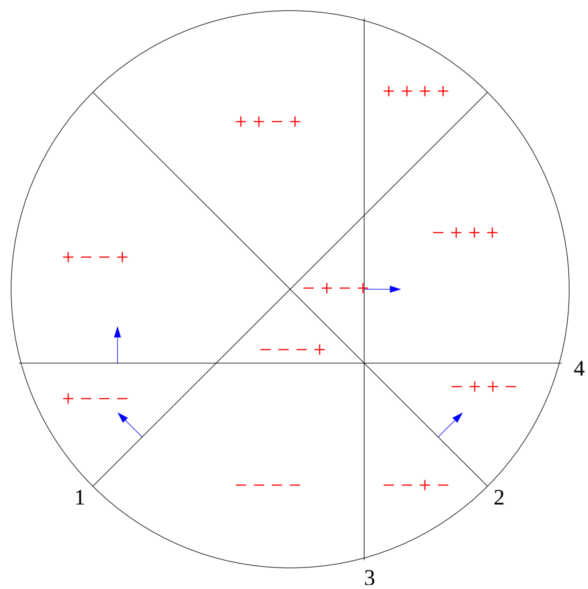
Example 1. Consider the central arrangement $\mathcal{A}$ whose hyperplanes are the zero sets of the linear forms

$$\ell_1 = x + y, \qquad \ell_2 = x - y, \qquad \ell_3 = x - z, \qquad \ell_1 = y + z.$$

The corresponding normal vectors are $V = \{\vec{v}_1, \ldots, \vec{v}_4\}$, where

$$\vec{v}_1 = (1, -1, 0), \qquad \vec{v}_2 = (1, 1, 0), \qquad \vec{v}_3 = (1, 0, 1), \qquad \vec{v}_4 = (0, 1, -1).$$

The projectivization $\operatorname{proj}(\mathcal{A})$ looks like this:



Each region F that borders the equator has a polar opposite $-F$ such that $c(-F) = -c(F)$.

The regions with covectors $---+$ and $-+-+$ do not border the equator, i.e., they are bounded in $\operatorname{proj}(\mathcal{A})$. Since they do not border the equator, neither do their opposites in $\mathcal{A}$, so those opposites do not occur in $\operatorname{proj}(\mathcal{A})$.

In the figure of Example 1, consider the point $p = \ell_2 \cap \ell_3 \cap \ell_4$. That three lines intersect at p means that there is a linear dependence among the corresponding normal vectors.

$$\vec{v}_2 - \vec{v}_3 + \vec{v}_4 = 0,$$

or on the level of linear forms,

$$\ell_2 - \ell_3 + \ell_4 = 0. \tag{1}$$

Of course, knowing which subsets of V are linearly dependent is equivalent to knowing the matroid M represented by V. Indeed, $\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a circuit of M.

However, (1) tells us more than that: there exists no $\vec{x} \in \mathbb{R}^3$ such that

$$\ell_2(x) > 0, \qquad \ell_3(x) < 0, \quad \text{and} \quad \ell_4(x) > 0.$$

That is, $\mathcal{A}$ has no covector of the form $*{+}{-}{+}$ (for any $* \in \{+,-,0\}$). We say that $0{+}{-}{+}$ is the corresponding **oriented circuit**.

For $c \in \{+,-,0\}^n$, write

$$c_+ = \{i \mid c_i = +\}, \qquad c_- = \{i \mid c_i = -\}.$$

Definition: Let n be a positive integer. A **circuit system for an oriented matroid** is a collection $\mathscr{C}$ of n-tuples $c \in \{+,-,0\}^n$ satisfying the following properties:

(1) $00\cdots 0 \notin \mathscr{C}$.
(2) If $c \in \mathscr{C}$, then $-c \in \mathscr{C}$.
(3) If $c, c' \in \mathscr{C}$ and $c \neq c'$, then it is not the case that both $c_+ \subset c'_+$ and $c_- \subset c'_-$
(4) If $c, c' \in \mathscr{C}$ and $c \neq c'$, and there is some i with $c_i = +$ and $c'_i = -$, then there exists $d \in \mathscr{C}$ with $d_i = 0$, and, for all $j \neq i$, $d_+ \subset c_+ \cup c'_+$ and $d_- \subset c_- \cup c'_-$.

Again, the idea is to record not just the linearly dependent subsets of a set $\{\ell_i, \ldots, \ell_n\}$ of linear forms, but also the sign patterns of the corresponding linear dependences, or "syzygies".

Condition (1) says that the empty set is linearly independent.

Condition (2) says that multiplying any syzygy by -1 gives a syzygy.

Condition (3), as in the definition of the circuit system of an (unoriented) matroid, must hold if we want circuits to record syzygies with minimal support.

Condition (4) is the oriented version of circuit exchange. Suppose that we have two syzygies

$$\sum_{j=1}^{n} \gamma_j \ell_j \;=\; \sum_{j=1}^{n} \gamma'_j \ell_j \;=\; 0,$$

with $\gamma_i > 0$ and $\gamma'_i < 0$ for some i. Multiplying by positive scalars if necessary (hence not changing the sign patterns), we may assume that $\gamma_i = -\gamma'_i$. Then

$$\sum_{j=1}^{n} \delta_j \ell_j \;=\; 0,$$

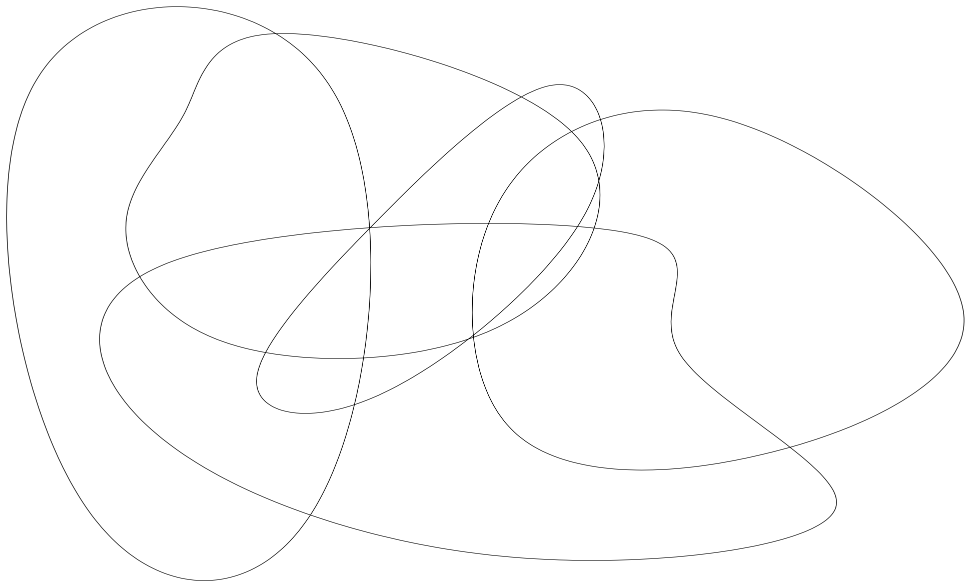
where $\delta_j = \gamma_j + \gamma'_j$. In particular, $\delta_i = 0$, and δ_j is positive (resp., negative) if and only if at least one of γ_j, γ'_j is positive (resp., negative).

- The set
$$\{c_+ \cup c_- \mid c \in \mathscr{C}\}$$
forms a circuit system for an (ordinary) matroid.

- Just as every graph gives rise to a matroid, any loopless directed graph gives rise to an oriented matroid (homework problem!)

As in the unoriented setting, the circuits of an oriented matroid represent minimal obstructions to being a covector. That is, for every real hyperplane arrangement $\mathcal{A}$, we can construct a circuit system $\mathscr{C}$ for an oriented matroid such that if k is a covector of $\mathcal{A}$ and c is a circuit, then it is not the case that $k_+ \supseteq c_+$ and $K_- \supseteq c_-$.

More generally, we can construct an oriented matroid from any real *pseudosphere arrangement*, i.e., a collection of homotopy $d-1$-spheres embedded in $\mathbb{R}^n$ such that the intersection of the closures of the spheres in any subcollection is connected or empty. Here is an example of a pseudocircle arrangement in $\mathbb{R}^2$:



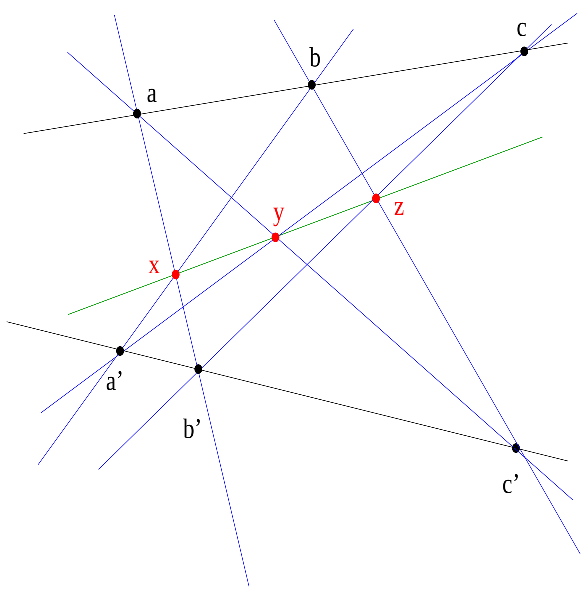
In fact, the Topological Representation Theorem of Folkman and Lawrence (1978) says that every oriented matroid can be realized by such a pseudosphere arrangement. However, there exist (lots of!) oriented matroids that cannot be realized as hyperplane arrangements.

Example: Pappus' Theorem from Euclidean geometry says the following:

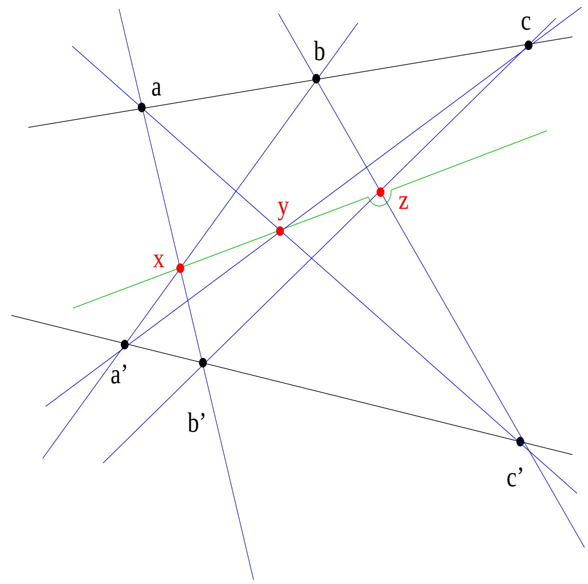
Let a, b, c, a', b', c' be distinct points in $\mathbb{R}^2$ such that a, b, c and a', b', c' are collinear. Then the three points

$$x = \overline{ab'} \cap \overline{a'b},$$
$$y = \overline{ac'} \cap \overline{a'c},$$
$$z = \overline{bc'} \cap \overline{b'c}$$

are collinear.



- If we perturb the green line a little bit so that it meets x and y but not z, we obtain a pseudoline arrangement whose oriented matroid $\mathcal{M}$ cannot be realized by means of a line arrangement.



- Pappus' Theorem can be proven using analytic geometry. The equations that say that x, y, z are collinear work over any field. Therefore, "unorienting" $\mathcal{M}$ produces a matroid that is not representable over any field.