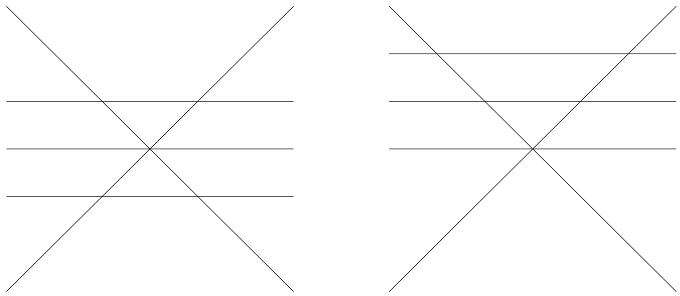
Monday 3/24

The Big Face Lattice

Let $\mathcal{A}, \mathcal{A}'$ be the following two affine line arrangements in $\mathbb{R}^2$. Are they isomorphic?



They have the same intersection poset (and therefore the same characteristic polynomial, which happens to be $k^2 - 5k + 6$) but non-isomorphic dual graphs—only the dual graph of $\mathcal{A}'$ has a vertex of degree 4.

Therefore, we'd like to have some notion of "isomorphism" of real hyperplane arrangements that distinguishes between these two.

Definition 1. Let $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{R}^d$ be a hyperplane arrangement, and let $\ell_1, \ldots, \ell_n$ be linear forms such that $H_i = \{\vec{x} \in \mathbb{R}^d \mid \ell_i(\vec{x}) = 0\}$. Let $c = (c_1, \ldots, c_n)$, where $c_i \in \{+, -, 0\}$ for each i. Consider the system of equations and inequalities

$$\begin{cases} \ell_i(\vec{x}) > 0 & \text{if } c_i = + \\ \ell_i(\vec{x}) < 0 & \text{if } c_i = - \\ \ell_i(\vec{x}) = 0 & \text{if } c_i = 0. \end{cases}$$

If the solution set of this system is nonempty, it is called a **face** of $\mathcal{A}$, and c is called a **covector**. The set of all faces is denoted $\mathscr{F}(\mathcal{A})$.

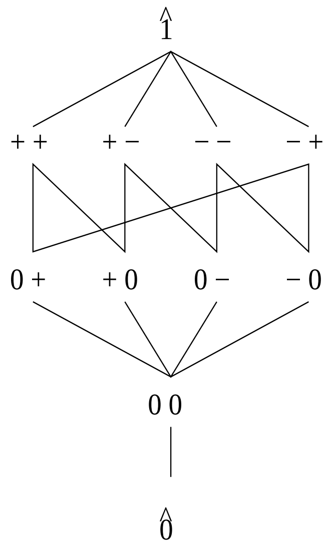
Example: Let $\mathcal{A} = \mathscr{B}_2$. Let H_1 and H_2 be the x- and y-axes respectively, so that we may take $\ell_i(x, y) = y$ and $\ell_2(x, y) = x$. The members of $\mathscr{F}(\mathcal{A})$ are as follows:

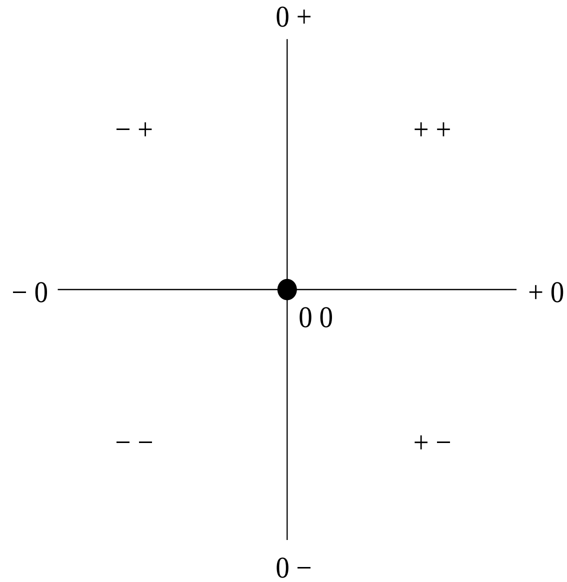
Name	Set	Covector
Origin	$\{(0,0)\}$	00
Positive x-axis	$\{(x,0) \mid x > 0\}$	$+0$
Negative x-axis	$\{(x,0) \mid x < 0\}$	-0
Positive y-axis	$\{(0,y) \mid y > 0\}$	$0+$
Negative y-axis	$\{(0,y) \mid y < 0\}$	$0-$
1st quadrant	$\{(x,y) \mid x > 0, y > 0\}$	$++$
2nd quadrant	$\{(x,y) \mid x < 0, y > 0\}$	$-+$
3rd quadrant	$\{(x,y) \mid x < 0, y < 0\}$	$--$
4th quadrant	$\{(x,y) \mid x > 0, y < 0\}$	$+-$

The set $\mathscr{F}(\mathcal{A})$ has a natural partial ordering, given by $F \leq F'$ whenever $\bar{F} \subseteq \bar{F}'$, where the bar denotes closure in the usual topology on $\mathbb{R}^d$. Equivalently, if c, c' are the covectors of F, F' respectively, then $c_i \in \{c'_i, 0\}$ for every i.

Proposition 1. *The partially ordered set $\hat{\mathscr{F}}(\mathcal{A}) = \mathscr{F}(\mathcal{A}) \cup \{\hat{0}, \hat{1}\}$ is a ranked lattice, called the **big face lattice** of $\mathcal{A}$.* (Note: The adjective "big" modifies "lattice", not "face".)

For example, the Hasse diagram of $\hat{\mathscr{F}}(\mathcal{B}_2)$ is shown on the left of the figure below. Since the Hasse diagram can be quite messy, it is typically more useful to draw a picture of $\mathcal{A}$ in which each face is labeled by its covector, as on the right.





If F is a face of $\mathcal{A}$ with covector c, then the affine span of F of $\mathcal{A}$ is an intersection of hyperplanes in $\mathcal{A}$, namely those for which $c_i = 0$. Therefore, we can recover the intersection poset $L(\mathcal{A})$ from $\mathscr{F}(\mathcal{A})$.

The coatoms of $\hat{\mathscr{F}}(\mathcal{A})$ are the regions of $\mathbb{R}^d \setminus \mathcal{A}$. The corresponding **maximal covectors** consist entirely of $+$'s and $-$'s, with no 0's. We can recover the dual graph of $\mathcal{A}$ from $\mathscr{F}(\mathcal{A})$, because two maximal covectors represent adjacent regions if and only if they differ in exactly one digit.