Wednesday 3/12

Modular Elements

Let L be a lattice. Recall that L is *modular* if it is ranked, and its rank function r satisfies

$$r(x) + r(y) = r(x \vee y) + r(x \wedge y) \tag{1}$$

for every $x, y \in L$. (This is not how we first defined modular lattices, but we proved that it is an equivalent condition; see notes from 1/30 and 2/1.)

Definition 1. An element $x \in L$ is a **modular element** if (1) holds for every $y \in L$. Thus L is modular if and only if every element of L is modular.

- The elements $\hat{0}$ and $\hat{1}$ are clearly modular in any lattice.

- If L is geometric, then every atom x is modular. Indeed, for $y \in L$, if $y \geq x$, then $y = x \vee y$ and $x = x \wedge y$, while if $y \not\geq x$ then $y \wedge x = \hat{0}$ and $y \vee x \gtrdot y$.

- The coatoms of a geometric lattice, however, need not be modular. Let $L = \Pi_n$; recall that Π_n has rank function $r(\pi) = n - |\pi|$. Let $x = 12|34$, $y = 13|24 \in \Pi_4$. Then $r(x) = r(y) = 2$, but $r(x \vee y) = r(\hat{1}) = 3$ and $r(x \wedge y) = r(\hat{0}) = 0$. So x is not a modular element.

Proposition 1. *The modular elements of Π_n are exactly the partitions with at most one nonsingleton block.*

Proof. Suppose that $\pi \in \Pi_n$ has one nonsingleton block B. For $\sigma \in \Pi_n$, let

$$X = \{C \in \sigma \mid C \cap B \neq \emptyset\}, \qquad Y = \{C \in \sigma \mid C \cap B = \emptyset\}.$$

Then

$$\pi \wedge \sigma = \Big\{C \cap B \mid C \in X\Big\} \cup \Big\{\{i\} \mid i \notin B\Big\},$$
$$\pi \vee \sigma = \left\{\bigcup_{C \in X} C\right\} \cup Y$$

so

$$\begin{aligned} |\pi \wedge \sigma| + |\pi \vee \sigma| &= (|X| + n - |B|) + (1 + |Y|) \\ &= (n - |B| + 1) + (|X| + |Y|) \;=\; |\pi| + |\sigma|, \end{aligned}$$

proving that π is a modular element.

For the converse, let B, C be nonsingleton blocks of π, then let σ have the two nonsingleton blocks $\{i, k\}, \{j, \ell\}$, where $i, j \in B$ and $k, \ell \in C$. Then $r(\sigma) = 2$ and $r(\pi \wedge \sigma) = r(\hat{0}) = 0$, but

$$r(\pi \vee \sigma) = r(\pi) + 1 < r(\pi) + r(\sigma) - r(\pi \wedge \sigma)$$

so π is not a modular element. □

The usefulness of a modular element is that if one exists, we can factor the characteristic polynomial of L.

Theorem 2. *Let L be a geometric lattice of rank n, and let $z \in L$ be a modular element. Then*

$$\chi_L(k) \;=\; \chi_{[\hat{0},z]}(k) \cdot \left[\sum_{y:\; y \wedge z = \hat{0}} \mu_L(\hat{0}, y) k^{n - r(z) - r(y)}\right]. \tag{2}$$

I'll skip the proof, which uses calculation in the Möbius algebra; see Stanley, HA, pp. 50–52.

Corollary 3. *Let L be a geometric lattice, and let $a \in L$ be an atom. Then*

$$\chi_L(k) \;=\; (k-1) \sum_{x:\; x \not\geq a} \mu_L(\hat{0},x) k^{r(L)-1-r(x)}.$$

(We already knew that $k-1$ had to be a factor of $\chi_L(k)$, because $\chi_L(1) = \sum_{x \in L} \mu_L(\hat{0},x) = 0$. Still, it's nice to see it another way.)

Corollary 4. *Let L be a geometric lattice, and let $z \in L$ be a coatom that is a modular element. Then*

$$\chi_L(k) \;=\; (k-e)\chi_{[\hat{0},z]}(k),$$

where e is the number of atoms $a \in L$ such that $a \not\leq z$.

Example 1. Corollary 4 provides another way of calculating the characteristic polynomial of Π_n. Let z be the coatom with blocks $[n-1]$ and $\{n\}$, which is a modular element by Proposition 1. There are $n-1$ atoms $a \not\leq z$, namely the partitions whose nonsingleton block is $\{i,n\}$ for some $i \in [n-1]$, so we obtain

$$\chi_{\Pi_n}(k) \;=\; (k-n+1)\chi_{\Pi_{n-1}}(k)$$

and by induction

$$\chi_{\Pi_n}(k) \;=\; (k-1)(k-2)\cdots(k-n+1).$$

Supersolvable Lattices

Let L be a geometric lattice with atoms A. Recall from (2) that if z is a modular element of L, then the characteristic polynomial of L factors:

$$\chi_L(k) \;=\; \chi_{[\hat{0},z]}(k) \cdot \left[\sum_{y:\; y \wedge z = \hat{0}} \mu_L(\hat{0},y) k^{n-r(z)-r(y)} \right].$$

Of course, we can always apply this for an atom z (Corollary 3). But, as we've seen with Π_n, something even better happens if z is a *coatom*: we can express $\chi_L(k)$ as the product of a linear form (the bracketed sum) with the characteristic polynomial of a smaller geometric lattice, namely $[\hat{0},z]$.

If we are extremely lucky, L will have a maximal chain of modular elements

$$\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{n-1} \lessdot x_n = \hat{1}.$$

In this case, we can apply Corollary 4 successively with $z = x_{n-1}$, $z = x_{n-2}$, $\dots$, $z = x_1$ to split the characteristic polynomial completely into linear factors:

$$\begin{aligned}
\chi_L(k) &= (k-e_{n-1})\chi_{[\hat{0},x_{n-1}]}(k) \\
&= (k-e_{n-1})(k-e_{n-2})\chi_{[\hat{0},x_{n-2}]}(k) \\
&= \dots \\
&= (k-e_{n-1})(k-e_{n-2})\cdots(k-e_0),
\end{aligned}$$

where

$$\begin{aligned}
e_i &= \#\{\text{atoms } a \text{ of } [\hat{0},x_{i+1}] \mid a \not\leq x_i\} \\
&= \#\{a \in A \mid a \leq x_{i+1},\; a \not\leq x_i\}.
\end{aligned}$$

Definition 2. A geometric lattice L is **supersolvable** if it has a *modular maximal chain*, that is, a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ such that every x_i is a modular element. A central hyperplane arrangement $\mathcal{A}$ is called supersolvable if $L(\mathcal{A})$ is supersolvable.

- Any modular lattice is supersolvable, because every maximal chain is modular.
- Π_n is supersolvable. because we can take x_i to be the partition whose unique nonsingleton block is $[i+1]$. Thus the braid arrangement Br_n is supersolvable.

- Let $G = C_4$ (a cycle with four vertices and four edges), and let $\mathcal{A} = \mathcal{A}_G$. Then $L(\mathcal{A})$ is the lattice of flats of the matroid $U_3(4)$; i.e.,

$$L = \{F \subseteq [4] : \ |F| \neq 3\}$$

with $r(F) = \min(|F|, 3)$. This lattice is not supersolvable, because no element at rank 2 is modular. For example, let $x = 12$ and $y = 34$; then $r(x) = r(y) = 2$ but $r(x \vee y) = 3$ and $r(x \wedge y) = 0$.

Theorem 5. *Let $G = (V, E)$ be a simple graph. Then $\mathcal{A}_G$ is supersolvable if and only if the vertices of G can be ordered $v_1, \ldots, v_n$ such that for every $i > 1$, the set*

$$C_i \ := \ \{v_j \mid j \leq i, \ v_i v_j \in E\}$$

forms a clique in G.

I'll omit the proof, which is not too hard; see Stanley, pp. 55–57. An equivalent condition is that G is a **chordal graph**: if $C \subseteq G$ is a cycle of length ≥ 4, then some pair of vertices that are not adjacent in C are in fact adjacent in G.

By the way, it is easy to see that if G satisfies the condition of Theorem 5, then the chromatic polynomial $\chi(G; k)$ splits into linear factors. Consider what happens when we color the vertices in order. When we color vertex v_i, it has $|C_i|$ neighbors that have already been colored, and they all have received different colors because they form a clique. Therefore, there are $k - |C_i|$ possible colors available for v_i, and we see that

$$\chi(G; k) \ = \ \prod_{i=1}^{n} (k - |C_i|).$$