Monday 3/10

Projectivization and Coning

Let K be a field. Denote points of K^n by $\vec{x} = (x_1, \dots, x_n)$.

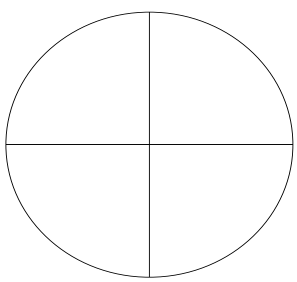
Projective space $\mathbb{P}^{n-1}K$ is by definition the set of lines through the origin in K. If $K = \mathbb{R}$, we can regard $\mathbb{P}^{n-1}\mathbb{R}$ as the unit sphere S^{n-1} with opposite points identified; in particular, it is an $(n-1)$-dimensional manifold.

Algebraically, write $\vec{x} \sim \vec{y}$ if $\vec{x}$ and $\vec{y}$ are nonzero scalar multiples of each other. Then $\sim$ is an equivalence relation, and

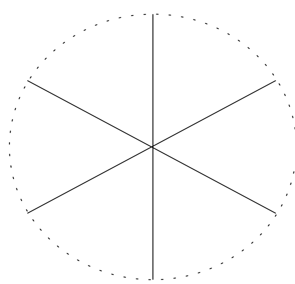
$$\mathbb{P}^{n-1}K = \left(K^n \setminus \{\vec{0}\}\right) / \sim .$$

Linear hyperplanes in K^n correspond to affine hyperplanes in $\mathbb{P}^{n-1}K$. Thus, given a central arrangement $\mathcal{A} \subset K^n$, we can construct its **projectivization** $\operatorname{proj}(\mathcal{A}) \subset \mathbb{P}^{n-1}K$.

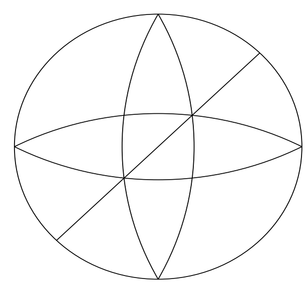
Projectivization supplies a nice way to draw central 3-dimensional real arrangements. Let S be the unit sphere, so that $H \cap S$ is a great circle for every $H \in \mathcal{A}$. Regard $H_0 \cap S$ as the equator and project the northern hemisphere into your piece of paper.



proj($\mathscr{B}_3$)

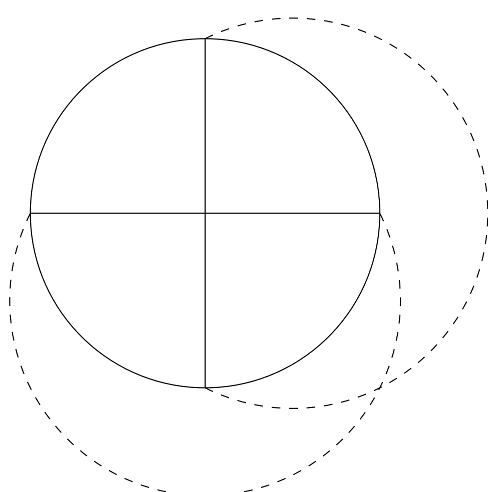


proj(Br_3)



proj(ess(Br_3))

Of course, a diagram of $\operatorname{proj}(\mathcal{A})$ only shows the "upper half" of $\mathcal{A}$. We can recover $\mathcal{A}$ from $\operatorname{proj}(\mathcal{A})$ by "reflecting the interior of the disc to the exterior" (Stanley). For example, when $\mathcal{A} = \mathscr{B}_3$:



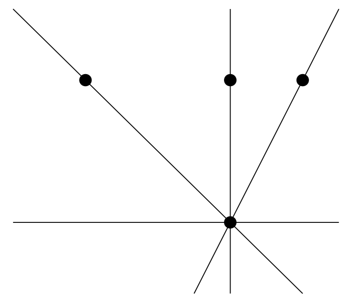
In particular, $r(\operatorname{proj}(\mathcal{A})) = \frac{1}{2} r(\mathcal{A})$.

Definition: Let $\mathcal{A} \subset K^n$ (not necessarily central). The **cone** $c\mathcal{A}$ is the central arrangement in K^{n+1} defined as follows:

- *Geometrically:* Make a copy of $\mathcal{A}$ in K^{n+1}, choose a point p not in any hyperplane of $\mathcal{A}$, and replace each $H \in \mathcal{A}$ with the affine span H' of p and H (which will be a hyperplane in K^{n+1}). Then, toss in one more hyperplane containing p and in general position with respect to every H'.
- *Algebraically:* For $H = \{\vec{x} \mid L(\vec{x}) = a_i\} \in \mathcal{A}$ (with L a homogeneous linear form on K^n and $a_i \in K$), construct a hyperplane $H' = \{(x_1, \ldots, x_n, y) \mid L(\vec{x}) = a_i y\} \subset K^{n+1}$ in $c\mathcal{A}$. Then, toss in the hyperplane $y = 0$.

For example, if $\mathcal{A}$ consists of the points $x = 0$, $x = -3$ and $x = 5$ in $\mathbb{R}^1$, then $c\mathcal{A}$ consists of the lines $x = y$, $x = -3y$, $x = 5y$, and $y = 0$ in $\mathbb{R}^2$.

$\mathcal{A}$



$c\mathcal{A}$

Proposition 1. $\chi_{c\mathcal{A}}(k) = (k-1)\chi_{\mathcal{A}}(k)$.

We'll prove this next time. In particular, Zaslavsky's formula for the number of regions implies that $r(c\mathcal{A}) = 2r(\mathcal{A})$.

More on Regions and the Characteristic Polynomial

Let $\mathcal{A} \subset K^n$ be a hyperplane arrangement. We have seen that if $K = \mathbb{R}$, then

$$r(\mathcal{A}) = \#\text{ of regions of } \mathcal{A} \;=\; (-1)^{\dim \mathcal{A}}\chi_{\mathcal{A}}(-1),$$
$$b(\mathcal{A}) = \#\text{ of rel. bounded regions} \;=\; (-1)^{\operatorname{rank} \mathcal{A}}\chi_{\mathcal{A}}(1).$$

(Zaslavsky's theorems) and that if $K = \mathbb{F}_q$, then

(1) $$|\mathbb{F}_q^n \setminus \mathcal{A}| \;=\; \chi_{\mathcal{A}}(q)$$

(a fact first noticed implicitly by Crapo and Rota and explicitly by Athanasiadis).

What if $\mathcal{A} \subset \mathbb{C}^n$ is a *complex* hyperplane arrangement? Since the hyperplanes of $\mathcal{A}$ have codimension 2 as real vector subspaces, the complement $X = \mathbb{C}^n \setminus \mathcal{A}$ is connected, but not simply connected.

Theorem 2 (Brieskorn 1971)**.** *The homology groups $H_i(X, \mathbb{Z})$ are free abelian, and the Poincáre polynomial of X is the characteristic polynomial backwards:*

$$\sum_{i=0}^{n} \operatorname{rank}_{\mathbb{Z}} H_i(X, \mathbb{Z}) q^i \;=\; (-q)^n \chi_{L(\mathcal{A})}(-1/q).$$

Orlik and Solomon (1980) strengthened Brieskorn's result by giving a presentation of the cohomology ring $H^*(X, \mathbb{Z})$ in terms of $L(\mathcal{A})$, thereby proving that the cohomology is a combinatorial invariant of $\mathcal{A}$. (Brieskorn's theorem says only that the additive structure of $H^*(X, \mathbb{Z})$ is a combinatorial invariant.)

The homotopy type of X is *not* a combinatorial invariant (according to Reiner, by a result of Rybnikov).

Graphic Arrangements

Definition 1. Let G be a simple graph on vertex set $[n]$. The **graphic arrangement** $\mathcal{A}_G \subset K^n$ consists of the hyperplanes $x_i = x_j$, where ij is an edge of G.

The arrangement $\mathcal{A}_G$ is central (but not essential), so $L(\mathcal{A}_G)$ is a geometric lattice. The corresponding matroid is naturally isomorphic to the graphic matroid of G. In particular, $r(\mathcal{A}_G) = T(G; 2, 0)$ equals the number of acyclic orientations of G.

For instance, if $G = K_n$, then $\mathcal{A} = Br_n$, which we have seen has $n!$ regions. On the other hand, the acyclic orientations of K_n are in bijection with total orderings of its vertices.

Moreover, the chromatic polynomial of G equals the characteristic polynomial of $L(\mathcal{A}_G)$.

This last fact has a concrete combinatorial interpretation. Regard a point $(x_1, \ldots, x_n) \in \mathbb{F}_q^n$ as a q-coloring of G that assigns color x_i to vertex i. Then the proper q-colorings are precisely the points of $\mathbb{F}_q^n \setminus \mathcal{A}_G$. The number of such colorings is $\chi(G; q)$ (the chromatic polynomial of G evaluated at q); on the other hand, by (1), it is also the characteristic polynomial $\chi_{\mathcal{A}_G}(q)$. Since $\chi(G; q) = \chi_{\mathcal{A}_G}(q)$ for infinitely many q (namely, all integer prime powers), the polynomials must be equal.

For some graphs (such as complete graphs and trees), the chromatic polynomial factors into linear terms. For others, it doesn't.

Example 1. Let $G = C_4$, a cycle with four vertices and four edges), and let $\mathcal{A} = \mathcal{A}_G$. Then $L(\mathcal{A})$ is the lattice of flats of the matroid $U_3(4)$; i.e.,

$$L = \{F \subseteq [4] : \ |F| \neq 3\}$$

with $r(F) = \min(|F|, 3)$. Since the Möbius function of an element of L depends only on its rank, it is easy to check that

$$\chi_L(k) \ = \ k^3 - 4k^2 + 6k - 3 = (k-1)(k^2 - 3k + k).$$

Multiplying by $k^{\dim \mathcal{A}_L - \operatorname{rank} \mathcal{A}_L} = k^{4-3}$ gives the characteristic polynomial of $\mathcal{A}_L$, which is the chromatic polynomial of C_4:

$$\chi_{C_4}(k) \ = \ k(k-1)(k^2 - 3k + k).$$

So the question arises: For which graphs does the chromatic polynomial factor into linear terms? More generally, for which arrangements $\mathcal{A}$ does the characteristic polynomial $\chi_{\mathcal{A}}(k)$ factor?