Wednesday 3/5

Regions of Hyperplane Arrangements

Let $\mathcal{A} \subset \mathbb{R}^n$ be a real hyperplane arrangement. The **regions** of $\mathcal{A}$ are the connected components of $\mathbb{R}^n \setminus \mathcal{A} = \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Each component is the interior of a (bounded or unbounded) polyhedron; in particular, it is homeomorphic to $\mathbb{R}^n$. We write

$$r(\mathcal{A}) = \text{ number of regions of } \mathcal{A}.$$

We'd also like to count the number of bounded regions. However, we must be careful, because if $\mathcal{A}$ is not essential then every region is unbounded. Accordingly, call a region **relatively bounded** if the corresponding region in $\operatorname{ess}(\mathcal{A})$ is bounded, and define

$$b(\mathcal{A}) = \text{ number of relatively bounded regions of } \mathcal{A}.$$

Note that $b(\mathcal{A}) = 0$ if and only if $\operatorname{ess}(\mathcal{A})$ is central.

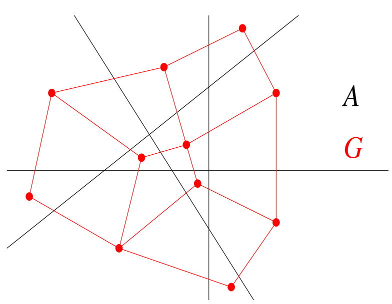
Example 1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the 2-dimensional arrangements shown on the left and right of the figure below, respectively.

Then

$$r(\mathcal{A}_1) = 6, \qquad b(\mathcal{A}_1) = 0, \qquad r(\mathcal{A}_2) = 10, \qquad b(\mathcal{A}_2) = 2.$$

Example 2. The Boolean arrangement $\mathscr{B}_n$ consists of the n coordinate hyperplanes in $\mathbb{R}^n$. The complement $\mathbb{R}^n \setminus \mathscr{B}_n$ is $\{(x_1, \ldots, x_n) \mid x_i \neq 0 \text{ for all } i\}$, and the connected components are the open orthants, specified by the signs of the n coordinates. Therefore, $r(\mathscr{B}^n) = 2^n$.

Example 3. Let $\mathcal{A}$ consist of m lines in $\mathbb{R}^2$ in *general position*: that is, no two lines are parallel and no three are coincident. Draw the *dual graph* G: the graph whose vertices are the regions of $\mathcal{A}$, with an edge between every two regions that share a common border.



Let

$$r = r(\mathcal{A}), \qquad v = \# \text{ of vertices of } G,$$
$$b = b(\mathcal{A}), \qquad e = \# \text{ of edges of } G,$$
$$f = \# \text{ of faces of } G.$$

Then

(1a) $$v = r,$$

(1b) $$f = 1 + \binom{m}{2} = \frac{m^2 - m + 2}{2}$$

(because each bounded region contains exactly one point where two lines of $\mathcal{A}$ meet); and

(1c) $$4(f-1) = 2e - (r-b)$$

(because each unbounded face has four sides).

(1d)

Note that the number $r - b$ of unbounded regions is just $2m$. (Take a walk around a very large circle. You will enter each unbounded region once, and will cross each line twice.) Therefore, from (1c) and (1b) we obtain

(1e) $$e = m + 2(f-1) = m^2.$$

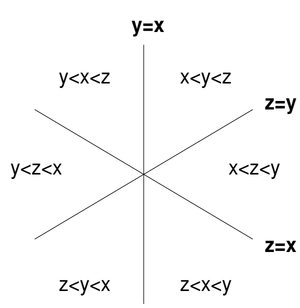
Now, Euler's formula for planar graphs says that $v - e + f = 2$. Substituting in (1a), (1b) and (1e) and solving for r gives

$$r = \frac{m^2 + m + 2}{2}$$

and therefore

$$b = r - 2m = \frac{m^2 - 3m + 2}{2} = \binom{m-1}{2}.$$

Example 4. The braid arrangement Br_n consists of the $\binom{n}{2}$ hyperplanes $x_i = x_j$ in $\mathbb{R}^n$. The complement $\mathbb{R}^n \setminus Br_n$ consists of all vectors in $\mathbb{R}^n$ with no two coordinates equal, and the connected components of this set are specified by the ordering of the set of coordinates as real numbers:



Therefore, $r(Br_n) = n!$

Our next goal is to prove Zaslavsky's theorems that the numbers $r(\mathcal{A})$ and $b(\mathcal{A})$ can be obtained as simple evaluations of the characteristic polynomial of the intersection poset $L(\mathcal{A})$.

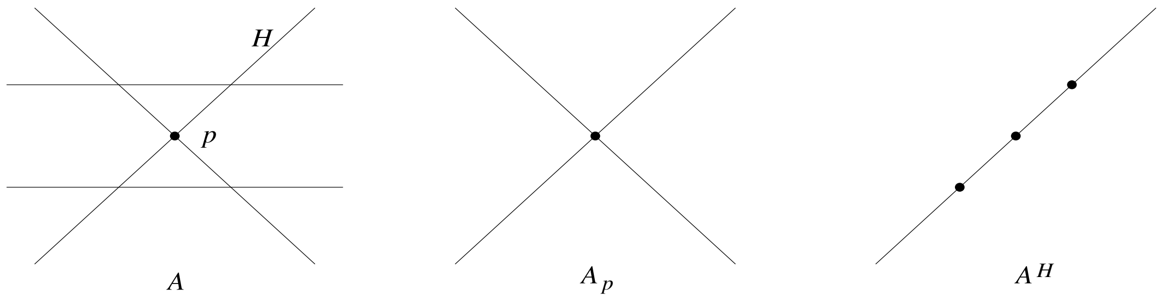
Deletion and Restriction

Let $\mathcal{A}$ be a hyperplane arrangement. If $\mathcal{A}$ is central, then we know that $L(\mathcal{A})$ is a geometric lattice; I'll write $M(\mathcal{A})$ for the corresponding matroid (represented, you will recall, by the normal vectors $\vec{n}_H$ to the hyperplanes $H \in \mathcal{A}$).

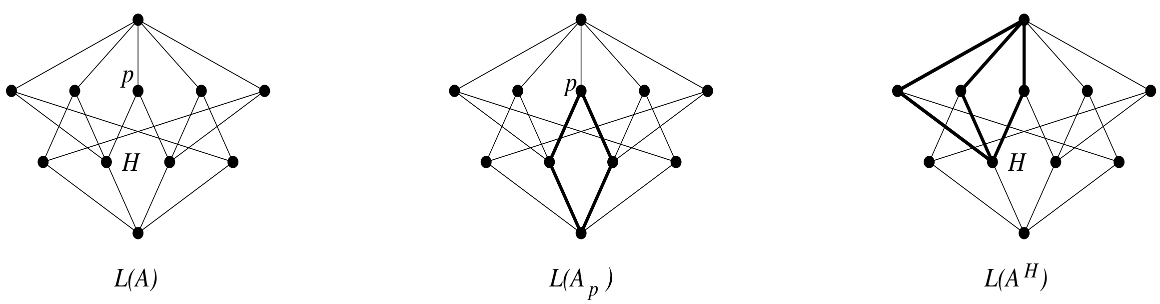
Let $x \in L(\mathcal{A})$. Recall that this means that x is an affine space formed by the intersection of some subset of $\mathcal{A}$. Define arrangements

$$\mathcal{A}_x = \{H \in \mathcal{A} \mid H \supseteq x\},$$
$$\mathcal{A}^x = \{W \mid W = H \cap x,\ H \in \mathcal{A} \setminus \mathcal{A}_x\}.$$

Example 5. Let $\mathcal{A}$ be the 2-dimensional arrangement shown on the left, with the line H and point p as shown. Then $\mathcal{A}_p$ and $\mathcal{A}^H$ are shown on the right.



The reason for this notation is that $L(\mathcal{A}_x)$ and $L(\mathcal{A}^x)$ are isomorphic respectively to the principal order ideal and principal order filter generated by x in $L(\mathcal{A})$.



We say that $\mathcal{A}^x$ is the *restriction* of $\mathcal{A}$ to x.

Notice that $\operatorname{rank} \mathcal{A}'$ equals either $\operatorname{rank} \mathcal{A} - 1$ or $\operatorname{rank} \mathcal{A}$, according as H is or is not a coloop in the matroid of $\mathcal{A}$, since $M(\mathcal{A}') = M(\mathcal{A}) - \vec{n}_H$.

Proposition 1. *Let $\mathcal{A}$ be a real arrangement and $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$. Then*

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'') \tag{2}$$

and

$$b(\mathcal{A}) = \begin{cases} b(\mathcal{A}') + b(\mathcal{A}'') & \textit{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}', \\ 0 & \textit{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}' + 1. \end{cases} \tag{3}$$

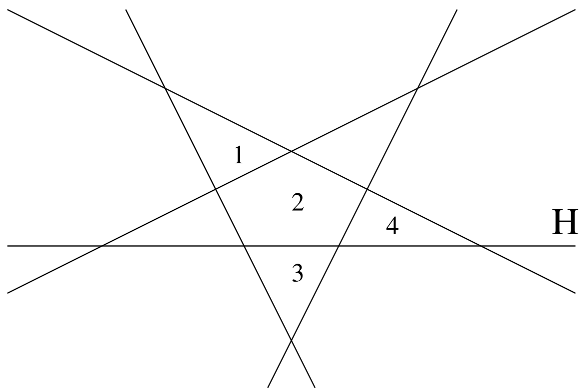
Proof. Consider what happens when we add H to $\mathcal{A}'$ to obtain $\mathcal{A}$. Some regions of $\mathcal{A}'$ will remain the same, while others will be split into two regions. The regions in the first category each count once in both $r(\mathcal{A})$ and $r(\mathcal{A}')$. The regions in the second category each contribute 2 to $r(\mathcal{A})$, but they also correspond bijectively to the regions of $\mathcal{A}''$. This proves (2).

By the way, if (and only if) H is a coloop then it borders *every* regionof $\mathcal{A}$, so $r(\mathcal{A}) = 2r(\mathcal{A}')$ in this case.

Now, what about bounded regions? If H is a coloop, then $\mathcal{A}$ has no bounded regions — every region of $\mathcal{A}'$ will contain a line, so every region of $\mathcal{A}$ will contain a ray. Otherwise, the bounded regions of $\mathcal{A}$ come in three flavors:

First, the regions not bordered by H (e.g., #1 below) correspond bijectively to bounded regions of $\mathcal{A}'$ through which H does not pass.

Second, for each region R of $\mathcal{A}$ bordered by H, the region $\bar{R} \cap H$ is bounded in $\mathcal{A}''$ (where $\bar{R}$ denotes the topological closure). Moreover, R comes from a bounded region in $\mathcal{A}'$ if and only if walking from R across H gets you to a bounded region of $\mathcal{A}$. (Yes in the case of the pair #2 and #3, which together contribute two to each side of (3); no in the case of #4, which contributes one to each side of (3).)



□

This looks a lot like a Tutte polynomial deletion/contraction recurrence. However, we only have a matroid to work with when $L(\mathcal{A})$ is a geometric lattice, that is, when $\mathcal{A}$ is central (otherwise, $L(\mathcal{A})$ is not even a bounded poset). On the other hand, $L(\mathcal{A})$ is certainly ranked (by codimension) for every arrangement, so we can work instead with its characteristic polynomial, which as you recall is defined as

$$\chi_{\mathcal{A}}(k) = \chi(L(\mathcal{A}); k) \;=\; \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) k^{\dim x}. \tag{4}$$

Proposition 2 (Deletion/Restriction)**.** *Let $\mathcal{A}$ be a real arrangement and $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}^H$. Then*

$$\chi_{\mathcal{A}}(k) \;=\; \chi_{\mathcal{A}'}(k) - \chi_{\mathcal{A}''}(k). \tag{5}$$

Proof coming next time!