Friday 2/15

More Matroid Constructions

2. Direct sum.

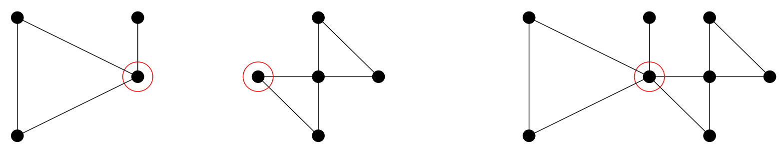
Let E_1, E_2 be disjoint sets, and let $\mathscr{B}_i$ be a basis system for a matroid M_i on E_i. The *direct sum* $M_1 \oplus M_2$ is the matroid on $E_1 \cup E_2$ with basis system

$$\mathscr{B} = \{B_1 \cup B_2 \mid B_1 \in \mathscr{B}_1,\ B_2 \in \mathscr{B}_2\}.$$

(I'll omit the routine proof that this is a basis system.)

If M_1, M_2 are linear matroids whose ground sets span vector spaces V_1, V_2 respectively, then $M_1 \oplus M_2$ is the matroid you get by regarding the vectors as living in $V_1 \oplus V_2$: the linear relations have to come either from V_1 or from V_2.

If G_1, G_2 are graphs with disjoint vertex sets, then $M(G_1) \oplus M(G_2) \cong M(G_1 + G_2)$, where $+$ denotes the disjoint union. Actually, something more is true: you can identify any vertex of G_1 with any vertex of G_2 and still get a graph whose associated graphic matroid is $M(G_1) \oplus M(G_2)$ (such as G in the following figure).



G_1 G_2 G

Corollary: Every graphic matroid arises from a *connected* graph.

Direct sum is additive on rank functions: for $A_1 \subseteq E_1$, $A_2 \subseteq E_2$, we have

$$r_{M_1 \oplus M_2}(A_1 \cup A_2) = r_{M_1}(A_1) + r_{M_2}(A_2).$$

The geometric lattice of a direct sum is a (Cartesian) product of posets:

$$L(M_1 \oplus M_2) \cong L(M_1) \times L(M_2),$$

subject to the order relations $(F_1, F_2) \leq (F_1', F_2')$ iff $F_i \leq F_i'$ in $L(M_i)$ for each i. (This is the operation you constructed in problem set #1, problem #2.)

As you should expect from an operation called "direct sum", and as the last two equations illustrate, pretty much all of the properties of $M_1 \oplus M_2$ can be deduced easily from those of its summands.

Definition 1. A matroid that cannot be written nontrivially as a direct sum of two smaller matroids is called *connected* or[‡] *indecomposable.*

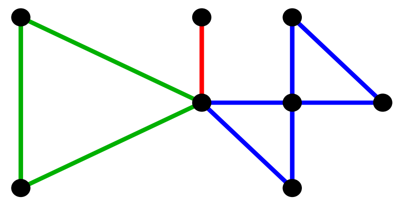
Proposition 1. *Let $G = (V, E)$ be a loopless graph. Then $M(G)$ is indecomposable if and only if G is* 2-connected*: not only is it connected, but it can't be disconnected by deleting a single vertex.*

The "only if" direction is immediate: the discussion above implies that

$$M(G) = \bigoplus_H M(H)$$

where H ranges over all the *blocks* (maximal 2-connected subgraphs) of G.

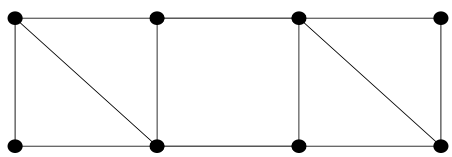
[‡]The first term is standard, but I prefer "indecomposable" to avoid potential confusion with the graph-theoretic meaning of "connected".

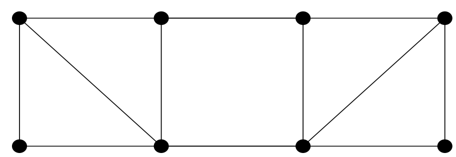


We'll prove the other direction later.

Remark: If $G \cong H$ as graphs, then clearly $M(G) \cong M(H)$. The converse is not true: if T is any tree (or even forest) on n vertices, then every set of edges is acyclic, so the independence complex is the Boolean algebra $\mathscr{B}_n$ (and, for that matter, so is the lattice of flats).

In light of Proposition 1, it is natural to suspect that every 2-connected graph is determined up to isomorphism by its graphic matroid, but even this is not true; the 2-connected graphs below are not isomorphic, but have isomorphic matroids.





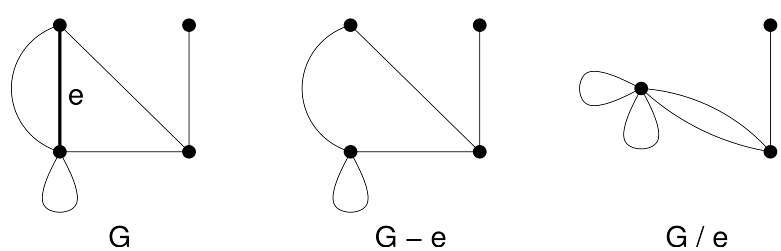
More on this later.

<u>3. Deletion and contraction.</u>

Definition 2. Let M be a matroid on E with basis system $\mathscr{B}$, and let $e \in E$.

(1) If e is not a coloop, then the *deletion* of e is the matroid $M - e$ (or $M \setminus e$) on $E \setminus \{e\}$ with bases
$$\{B \mid B \in \mathscr{B},\ e \notin B\}.$$
(2) If e is not a loop, then the *contraction* of e is the matroid M/e (or $M : e$) on $E \setminus \{e\}$ with bases
$$\{B \setminus \{e\} \mid B \in \mathscr{B},\ e \in B\}.$$

Again, the terms come from graph theory. Deleting an edge of a graph means what you think it means, while contracting an edge means to throw it away and to glue its endpoints together.



Notice that contracting can cause some edges to become parallel, and can cause other edges (namely, those parallel to the edge you're contracting) to become loops. In matroid language, deleting an element from a simple matroid always yields a simple matroid, but the same is not true for contraction.

How about the linear setting? Let V be a vector space over a field $\mathbb{F}$, let $E \subset V$ be a set of vectors with linear matroid M, and let $e \in E$. Then $M - e$ is just the linear matroid on $E \setminus \{e\}$, while M/e is what you get by projecting $E \setminus \{e\}$ onto the quotient space $V/(\mathbb{F}e)$. (For example, if e is the i^{th} standard basis vector, then erase the i^{th} coordinate of every vector in $E \setminus \{e\}$.)

Deletion and contraction are reversed by duality:

$$(M-e)^* \cong M^*/e \quad \text{and} \quad (M/e)^* \cong M^* - e.$$

Example: If M is the uniform matroid $U_k(n)$, then $M-e \cong U_k(n-1)$ and $M/e \cong U_{k-1}(n-1)$ for every ground set element e.

Many invariants of matroids can be expressed recursively in terms of deletion and contraction. The following fact is immediate from Definition 2.

Proposition 2. *Let M be a matroid on ground set E, and let $b(M)$ denote the number of bases of M. For every $e \in E$, we have*

$$b(M) = \begin{cases} b(M-e) & \text{if } e \text{ is a loop;} \\ b(M/e) & \text{if } e \text{ is a coloop;} \\ b(M-e)+b(M/e) & \text{otherwise.} \end{cases}$$

Example: If $M \cong U_k(n)$, then $b(M) = \binom{n}{k}$, and the recurrence of Proposition 2 is just the Pascal relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n}{k-1}.$$

This observation is the tip of an iceberg called the *Tutte polynomial* of a matroid.