Math 725, Spring 2010
Problem Set #7
Due date: Tuesday, May 4

#1. Verify the cut-edge case of Theorem 1 in the April 15 lecture notes on the Tutte polynomial (available from the course website). That is, prove that if e is a cut-edge, then

$$\tilde{T}(G;\, x, y) \;=\; x \cdot \tilde{T}(G/e;\, x, y),$$

where $\tilde{T}$ is the corank-nullity form of the Tutte polynomial.

#2. [West 6.1.21] Let G be a connected planar graph and let $A \subseteq E(G)$. Prove that A is a spanning tree of G if and only if $E(G^*) \setminus A^*$ is a spanning tree of G^*.

#3. Let G be a connected planar graph. Prove that

$$T(G;\, x, y) = T(G^*;\, y, x).$$

(Hint: There are two ways to do this. You can use the closed formula for the Tutte polynomial and compare the rank of a subset of $E(G)$ to its dual subset of $E(G^*)$, for which results like Problem #3 above and Theorem 6.1.14 may be helpful. Alternately, show that $(G-e)^* = G^*/e^*$ and $(G/e)^* = G^* - e^*$, and that taking the planar dual interchanges loops and cut-edges, then apply the recurrence.)

#4. A *regular polyhedron* (or *Platonic solid*) is a polyhedron in which all vertices have the same number of neighbors, all edges have the same length, all edge-edge angles are equal, and all face-face angles are equal—in other words, as symmetric as it can possibly be. (The most familiar example is probably the cube.) Classifying the Platonic solids was one of the big achievements of classical Greek geometry (although they didn't have graph theory as we know it).

Let G be the graph of a regular polyhedron P. Then G is a planar graph, because we can embed G on the surface of a sphere, then puncture the sphere to obtain a plane. Meanwhile, G is k-regular for some $k \geq 3$, and G^* is ℓ-regular for some $\ell \geq 3$.

Using handshaking, dual handshaking (i.e., that the sum of lengths of all faces equals $2e(G)$ for any planar graph), and Euler's formula, determine all possibilities for k and ℓ, and thus for the numbers of vertices, edges and faces of P.

This is the first step in classifying the Platonic solids. If you feel like it (for extra credit), try to go to the next step, i.e., showing that for each list (k, ℓ, n, e, f), there is (up to isomorphism) only one vertex-transitive graph, hence at most one Platonic solid, with these parameters.

#5. [West 6.2.12, modified] The two classical characterizations of planarity are:

Kuratowski's Theorem. A graph G is planar if and only if it has no $K_{3,3}$- or K_5-subdivision. (i.e., a subgraph of G isomorphic to a subdivision of $K_{3,3}$ or K_5.

Wagner's Theorem. A graph G is planar if and only if it has no $K_{3,3}$- or K_5-minor (i.e., a graph obtained from G by some sequence of vertex deletions, edge deletions, and edge contractions).

(a) Prove that deletion and contraction of edges preserve planarity. I.e., if G is planar then so are $G - e$ and G/e for every $e \in E(G)$. Using this, prove the "only if" direction of Wagner's Theorem.

[Hint: Deletion is easy, but contraction will require a bit of thought about the geometry. Using Tutte's and/or Fáry's Theorems may make life easier; e.g., it is OK to start with a straight-line embedding of G and construct an arbitrary planar embedding of G/e.]

(b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient. [I.e., prove that if G satisfies Wagner's condition of having no K_5- or $K_{3,3}$-minor, then it has no K_5- or $K_{3,3}$-subdivision, hence is planar by Kuratowski's theorem.]