Math 725, Spring 2010
Problem Set #6
Due date: Tuesday, April 20

#1. Menger's theorem can be extended to vertex *sets*. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$ and $[A, B] = \emptyset$, an *A, B-path* is a path with one endpoint in A and the other in B; an *A, B-cut* is a set of vertices $S \subseteq V(G) \setminus \{A, B\}$ such that $G - S$ contains no A, B-path; and a family $P_1, \ldots, P_n$ of A, B-paths is *pairwise disjoint* (or a *PD-family*) if $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$. (Note that even the endpoints are not allowed to coincide!) Prove that the minimum size of an A, B-cut equals the maximum size of a PD-family of A, B-paths. (Hint: Use the single-vertex version of Menger's theorem.)

#2. Derive the König-Egerváry Theorem from the Max-Flow/Min-Cut Theorem.

#3. [West 5.3.4] (a) Prove that the chromatic polynomial of the n-cycle is $p(C_n; k) = (k-1)^n + (-1)^n(k-1)$.

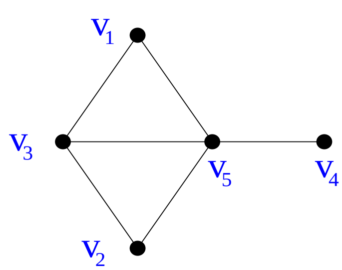
(b) For $H = G \vee K_1$, prove that $p(H; k) = k \cdot p(G; k-1)$. (Here $\vee$ denotes join; see Defn. 3.3.6 on p. 138.) Combine this with (a) to find the chromatic polynomial of the wheel $W_n = C_n \vee K_1$.

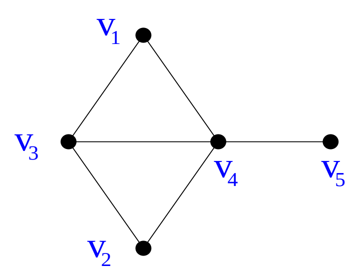
#4. Construct a graph G in which *no* ordering of the vertices $v_1, \ldots, v_n$ from highest to lowest degree produces an optimal coloring under the greedy algorithm. That is, if $d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$, then the greedy-coloring algorithm uses strictly more than $\chi(G)$ colors. Include a proof (which should not be difficult) of why your example satisfies these conditions. (I can think of an example with eight vertices; there may be a smaller one.)

(The idea of this problem is to show that while ordering the vertices by degree is *often* a good idea and, as we have seen, produces the best general bound on $\chi(G)$, it is not *necessarily* the best way to color a graph.)

#5. A graph G is called a **threshold graph** if its vertices can be ordered $v_1, \ldots, v_n$ so that for each i, $2 \leq i \leq n$, either v_i is adjacent to all or none of the vertices $v_1, \ldots, v_{i-1}$. For instance, the graph on the left is a threshold graph because

$$N(v_3) \supseteq \{v_1, v_2\}; \qquad N(v_4) \cap \{v_1, v_2, v_3\} = \emptyset; \qquad N(v_5) \supseteq \{v_1, v_2, v_3, v_4\}.$$





Warning: Being threshold says that there *exists* such an ordering. The graph on the right is isomorphic to the one on the left, but the given ordering of vertices is not a threshold ordering because $\emptyset \subsetneq N(v_5) \subsetneq \{v_1, v_2, v_3, v_4\}$.

(a) Prove that every threshold graph is chordal. (You can do this in either of two ways: either modify the threshold ordering to obtain an SEO, or prove directly that every cycle of length ≥ 4 has a chord. If the latter, be careful not to assume that the order of vertices in a cycle has anything to do with the threshold ordering.)

(b) Show that P_4 is not threshold (although it is certainly chordal).