Math 725, Spring 2010
Problem Set #2
Due date: Tuesday, February 9

#1. Problem 1.2.26 in West.

#2. Let G be a simple graph with n vertices and e edges. The *clique number* $\omega = \omega(G)$ is the largest number of vertices in a clique in G. (For example, $\omega(C_3) = 3$, but $\omega(C_n) = 2$ for $n \geq 4$.) Prove that if G is not a complete graph, then

$$e \leq \frac{(\omega+2)(\omega-1)n(n-1)}{2\omega(\omega+1)}.$$

#3. (a) Let $k \geq 2$ and let G be a k-regular bipartite graph. Prove that G has no cut-edge. (Hint: Let e be a cut-edge and let G' be one of the components of $G - e$. Count the edges of G' in two ways.)
(b) Construct a connected nonbipartite 3-regular graph with a cut-edge. (This shows that the condition "bipartite" really is necessary in (a).)

#4. Problem 1.2.29 in West.

#5. A *caterpillar* is a tree in which some path (the "spine") either contains or is incident to every edge. (See p. 88.) Let T be a tree with ℓ leaves. Prove that T is a caterpillar if and only if its diameter is $n - \ell + 1$.

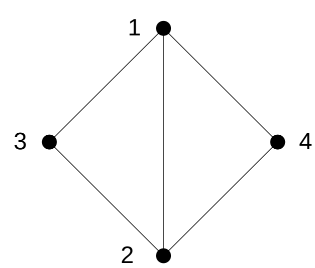
#6. Problem 2.1.3.7 in West. (This is called the *symmetric exchange property* for spanning trees.)

#7. Let G be a graph and fix an (arbitrary) orientation. The *signed incidence matrix* M of G is constructed as follows: Choose an orientation D of G. Build a matrix M with a column for each edge of G and a row for each vertex, and entries

$$m_{v,e} = \begin{cases} +1 & \text{if } v = \text{tail}(e), \\ -1 & \text{if } v = \text{head}(e), \\ 0 & \text{otherwise.} \end{cases}$$

(This is the same thing as an incidence matrix of a digraph; see Definition 1.4.10, p.56.) For each edge $e \in E(G)$, let M_e denote the column of M corresponding to e. For each $A \subseteq E(G)$, let $M_A = \{M_e \ : \ e \in A\}$.

For example, let G be the graph shown below.



(a) Pick an orientation of D and write out the corresponding signed incidence matrix.
(b) For which edge sets $A \subseteq E(G)$ does M_A form a basis for the column space of M?
(c) For which A is M_A linearly independent?
(d) For which sets A does M_{13} belong to the linear span of M_A?
(e) Do your answers depend on the choice of orientation?
(f) Do your answers change if you delete all the minus signs?
(g) Make as many conjectures you can about how the linear-algebraic properties of M correspond to the graph-theoretic properties of G for all graphs G.