Problem: Find a maximum matching in a given X, Y-bigraph G.

- **König-Egerváry Theorem:** $\alpha'(G) = \beta(G)$, where α' = size of a maximum matching, β = size of a minimum vertex cover

(In general $\alpha' \leq \beta$ ("weak duality"), with equality ("strong duality") for bipartite graphs).

- **Berge's Theorem:** A matching M is maximum if and only if G has no M-augmenting path.

Given a matching M, construct an orientation $D = D(M)$ of G by orienting each edge of M from Y to X, and orienting each edge of $E(G) - M$ from X to Y.

M-alternating path in G = path in D

M-augmenting path in G = path in D from $X - V(M)$ to $Y - V(M)$

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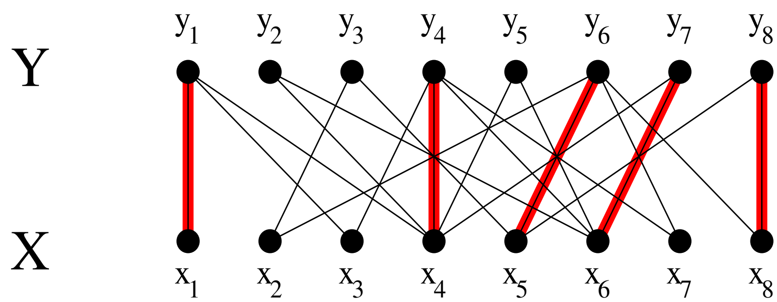
Augmenting Path Algorithm
Input:  an X,Y-bigraph G and a matching M

repeat
{
    X' := X − V(M)
    Y' := Y − V(M)
    A := {all vertices reachable from X' in D}
    If A ∩ Y' is nonempty, we have an augmenting path P;
        set M = M △ P
}
until A = ∅
```

We can compute A using Dijkstra's algorithm (really just breadth-first search). Essentially, compute the set of all vertices of D reachable from a vertex in $X - V(M)$.

(You can use Dijkstra's algorithm itself (for digraphs) to find the vertices of D reachable from each M-unsaturated vertex in X, one by one, then take the union of all such vertex sets found. This can be made more efficient—see West for details.)

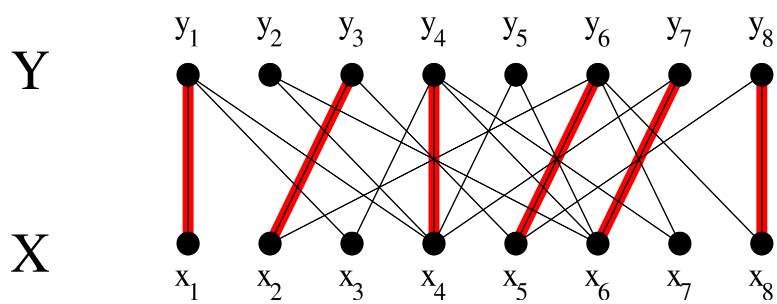
Example. Suppose that M starts out like this.



$X' = \{x_2,\ x_3,\ x_7\}$; $\qquad Y' = \{y_2,\ y_3,\ y_5\}$

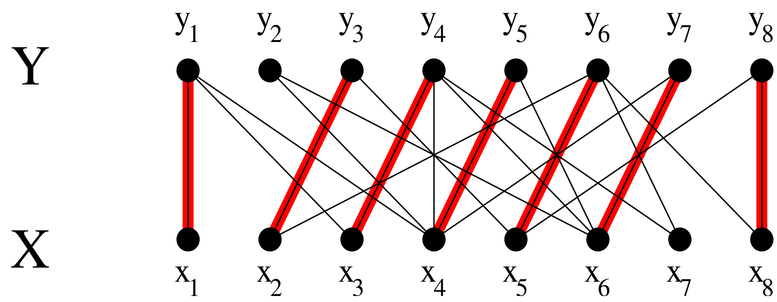
$y_3 \in A$. P here is one edge: $P = x_2, y_3$

So M is not maximum. Replace it with $M \triangle P$ (in this case, $M \cup P$):



$X' = \{x_3,\ x_7\}$; $\qquad Y' = \{y_2,\ y_5\}$

Again, BFS will turn up an augmenting path, possibly $P = x_3, y_4, x_4, y_5$.



At this point, BFS will not find an augmenting path and the algorithm terminates, with

$$A = \underbrace{\{x_1,\ x_2,\ x_3,\ x_5,\ x_7,\ x_8\}}_{A_X} \cup \underbrace{\{y_1,\ y_3,\ y_4,\ y_6,\ y_8\}}_{A_Y}$$

Every vertex in A_X has all of its neighbors in A_Y, i.e.,

$$N(A_X) \subset A_Y.$$

That is,

$$Q = A_Y \cap (X - A_X)$$

is a vertex cover.

Every vertex of Q is M-saturated. (If $y \in A_Y$ is unsaturated then the path to it is M-augmenting, and A_X contains all the unsaturated vertices of X.)

No edge of M has more than one endpoint in Q. (If $y \in A_Y$ then $\text{sp}(y) \in A_X$.)

Therefore $|Q| \leq |M|$.