Math 724, Fall 2017
Take-Home Test #3
Deadline: Wednesday, December 13, 10:00am

Instructions: Typeset your solutions in LaTeX. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name, e.g., `MirzakhaniTest3.pdf`. You may refer to the textbook and your class notes, and you may cite the result of any problem from previous assignments, or done in class. You may also use a computer algebra system such as Sage to carry out calculations and test conjectures. However, *you are not allowed to collaborate*; you may not consult any external resource or any human other than Jeremy.

Problem #1 The game of *bridge* is an advanced version of egdirb. It uses a standard deck with four suits (spades, hearts, diamonds, clubs), each with 13 cards (ace, king, queen, jack, 10, 9, ..., 2). There are four players. Each player is dealt a hand of 13 cards.

(#1a) [10 pts] Bridge players call a hand "balanced" if it contains at least 2 cards in every suit, and no more than 8 cards in any two suits. How many possible bridge hands are balanced?
(#1b) [10 pts] How many possible bridge hands contain at least one ace, at least one king, and at least one queen?

Problem #2 [20 pts] How many ways are there of making change for a three-dollar bill with pennies, nickels, dimes, and quarters that use at least one, but no more than ten, of each kind of coin? (The answer is **not** "Zero; there is no such thing as a three-dollar bill." Pretend there is.)

Problem #3 [20 pts] For a graph $G = (V, E)$, let $\phi(G)$ be the number of forests in G — that is, subsets of E that contain no cycles. For example, if G is itself a forest then $\phi(G) = 2^{|E|}$, and if G is a cycle then $\phi(G) = 2^{|E|} - 1$. Prove that

$$\phi(G) = \begin{cases} \phi(G-e) & \text{if } e \text{ is a loop,} \\ 2\phi(G/e) & \text{if } e \text{ is a bridge,} \\ \phi(G-e) + \phi(G/e) & \text{otherwise.} \end{cases}$$

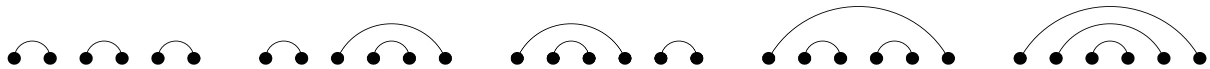
(A *loop* is an edge whose endpoints are equal; a *bridge* is an edge such that deleting it causes one component to split into two.)

Problem #4 [10 pts] Give a combinatorial interpretation for the coefficient of $x^n q^k$ in the power series expansion of the infinite product

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-qx^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-qx^4}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-qx^6}\right)\cdots.$$

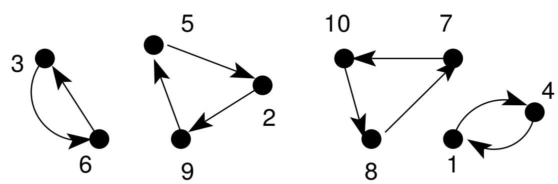
In other words, describe a set of combinatorial objects whose cardinality is the coefficient of $x^n q^k$.

Problem #5 [20 pts] A *noncrossing matching* consists of $2n$ points arranged in a line, together with n arcs linking the points in pairs, such that no two arcs cross. For example, the noncrossing matchings for $n = 3$ are shown below.



Prove that for all n, the number of noncrossing matchings on $2n$ points is the Catalan number C_n. (You can do this either by exhibiting a bijection to a set known to be counted by the Catalan numbers, or by verifying that the Catalan recurrence holds.)

Problem #6 Let $G(n)$ be the set of directed graphs on vertex set $[n]$ in which every component is either a pair of opposite edges, or a 3-cycle with edges oriented cyclically. For example, an element of $G(10)$ is shown below. Let $g(n) = |G(n)|$. By convention, we will set $g(0) = 1$, and $g(n) = 0$ for $n < 0$.



(#6a) [10 pts] What are the numbers $g(1), \ldots, g(5)$?

(#6b) [10 pts] Find a recurrence for $g(n)$. (Hint: Consider the cycle containing vertex n — there are two cases.)

(#6c) [10 pts] Let $y = \sum_{n=0}^{\infty} g(n)x^n/n!$ be the EGF for g. Translate the recurrence you just found into a differential equation for y.

(#6d) [10 pts] Solve the differential equation (it should not be hard) to obtain a nice closed form for y.

(#6e) [20 pts] Show that the EGF for the set of directed graphs on vertex set $[n]$ in which every component is an oriented cycle of *odd* length is

$$\sqrt{\frac{1+x}{1-x}}.$$