Math 724, Fall 2017
Take-Home Test #1
Deadline: Friday, September 22, 5:00pm

Instructions: Typeset your solutions in LaTeX. Email your solutions to Jeremy (jlmartin@ku.edu) as a PDF file named with your last name, e.g., `NoetherTest1.pdf`. You may refer to the textbook and your class notes, and you may cite the result of any problem from Chapter 1 assigned on HW #1 or #2 or done in class. However, unlike the homework, *you are not allowed to collaborate*; you may not consult any external resource or any human other than Jeremy.

Problem #1 In a bridge deal, each of 4 players (North, South, West and East) is dealt a hand of 13 cards from a standard deck of 52 cards.

(#1a) [5 pts] How many bridge hands contain exactly four spades?

(#1b) [10 pts] How many bridge hands contain more spades than hearts?

(#1c) [5 pts] How many different possible deals are there? (The four players are considered distinct. For example, interchanging North's hand with South's hand results in a different deal.)

Problem #2 [10 pts] Recall from Supplementary Problem 1 that a composition is an expression $n = a_1 + \cdots + a_k$, where the a_i are positive integers. A *weak composition* is the same thing, except that the a_i's are only required to be nonnegative rather than positive. Count the weak compositions of n into k parts.

Problem #3 [20 pts] Give a combinatorial proof of the identity

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \forall n \in \mathbb{N}.$$

(Half credit for giving an algebraic proof by induction.)

Problem #4 [20 pts] Let n be an integer not divisible by 2 or 5. Prove that there is some multiple of n in which every digit is 9. (For example, $n = 239$ is a divisor of 9999999.)

Problem #5 [10 pts] A *standard tableau of shape* $2 \times n$ is a $2 \times n$ grid filled with the numbers $1, \ldots, 2n$, using each number once, so that every row increases left to right and every column increases top to bottom. For example, there are five standard tableaux[1] of shape 2×3:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

Prove that for all $n > 0$, the number of standard tableaux of shape $2 \times n$ is the Catalan number C_n.

Problem #6 [20 pts] Let $m \geq n \geq 0$ be integers. Let $C(m, n)$ be the number of lattice paths from $(0, 0)$ to (m, n) that do not go above the line $y = x$. (For example, if $m = n$, then $C(m, n)$ is just the Catalan number C_n.) Find a simple formula for $C(m, n)$ that generalizes the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. (Hint: Generalize the method of problem 51 in the textbook.)

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[1] "Tableau" is singular; "tableaux" is plural.