Math 724, Fall 2013
Take-Home Test #2

Instructions: Write up your solutions using LaTeX. You may use books and notes, but you are not allowed to collaborate — you may not consult any human other than the instructor. Solutions are due at the start of class on **Friday, November 15.**

Problem #1 Let $n > 0$ be an integer.

(#1a) [10 pts] How many labeled trees T on vertex set $[n]$ have the property that the degree of every vertex is either 1 or 3? Your answer should be a function of n expressed without summation notation.

(#1b) [10 pts] Let k be an integer with $0 \leq k \leq n$. How many labeled trees on vertex set $[n]$ have the property that vertices $1, 2, 3, \ldots, k$ are all leaves (i.e., each shares an edge with exactly one other vertex)? (The tree can have other leaves as well.) Your answer should be a function of n and k expressed without summation notation.

Problem #2 [20 pts] Let $S(k, n)$ denote Stirling numbers of the second kind. Give a combinatorial proof that

$$S(k, n) = \sum_i \binom{k-1}{i-1} S(k-i, n-1)$$

for all positive integers k, n. (By "combinatorial," I mean "explain why both sides of the equation count the same set of objects" — do not give a purely algebraic proof using, say, induction.)

Problem #3 Give combinatorial interpretations for the following numbers. (In other words, describe what they count.)

(#3a) [10 pts] The coefficient of x^k in the infinite product

$$\prod_{n=1}^{\infty} (1 + x^n + x^{2n} + \cdots + x^{n^2}).$$

(#3b) [10 pts] The coefficient of $q^\ell x^k$ in the infinite product

$$\prod_{n=1}^{\infty} \frac{1}{1 - qx^n}.$$

Problem #4 [20 pts] Let p, q be positive integers and let $C(p, q)$ denote the set of weak compositions of p with q parts. Give an explicit bijection $C(p, q) \to C(q-1, p+1)$.

Problem #5 [20 pts] Recall that 1 Galleon is worth 17 Sickles and 1 Sickle is worth 29 Knuts. Suppose that the Ministry introduces a 3-Sickle and a 6-Knut piece (known respectively as a Trickle and a Hexknut). With the new coinage, how many ways are there of making change for a Galleon? (If you are not an expert at Arithmancy, I recommend that you use Sage or another computer algebra system to do the calculation.)