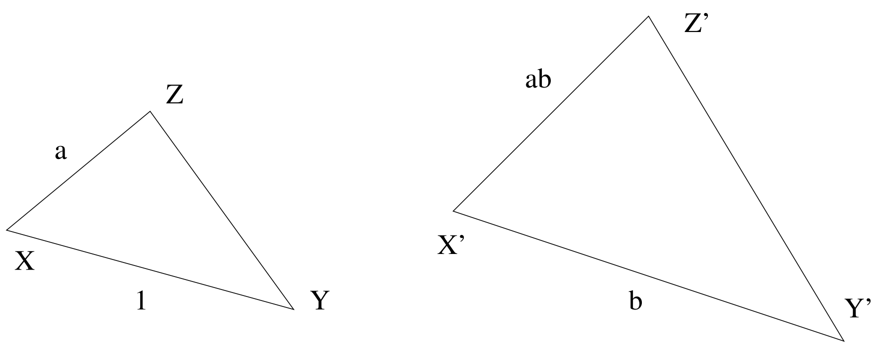
The three classical geometric problems

1 Constructible numbers

Suppose that you are given a line segment of length 1, and the Euclidean tools of compass and (unmarked) straightedge. What other lengths can you construct?

It's easy to construct other integer lengths n, by attaching n copies of your unit length back to back along a line. A little more generally, if you can construct lengths x and y, then you can construct $x+y$ and $x-y$.

What about multiplication and division? The trick is to use similar triangles. Given lengths a and b, you can construct ab as follows. First, draw a triangle ΔXYZ with $XY = 1$ and $XZ = a$. Then, draw a segment $X'Y'$ of length b, and construct a new triangle $\Delta X'Y'Z'$ similar to the first one. Then $X'Z' = ab$.



If instead you start by making $X'Z' = b$, then the new triangle will have $X'Y' = b/a$.

So the set $\mathbb{K}$ of all constructible numbers is closed under the four arithmetic operations of addition, subtraction, multiplication and division.[1] In particular, every rational number is constructible. (But there are certainly non-rational numbers that are constructible. For example, $\sqrt{2}$ can be constructed as the length of the hypotenuse of an isosceles right triangle. So can $\sqrt{n}$ for any integer n, in fact.

So the question arises: What exactly is the set $\mathbb{K}$? That is, which real numbers can be constructed as the lengths of line segments?

2 The three classical problems

There are three famous problems of classical geometry that the Greeks were unable to solve (for good reason). They were:

1. **Squaring the circle.** Given a circle, construct a square of the same area.
2. **Trisecting the angle.** Given an arbitrary angle of measure α, construct an angle of measure $\alpha/3$.
3. **Squaring the circle.** Given a cube, construct a new cube whose volume is double that of the first cube.

All these problems can be rephrased in terms of constructible numbers. A circle of radius 1 has area π, so we can square the circle iff we can construct $\sqrt{\pi}$. A cube Q of side length 1 has volume 1; to double it, we'd need to construct a line segment of length $\sqrt[3]{2}$. Finally, it turns out that trisecting an arbitrary angle is equivalent to being able to construct a root of a certain cubic equation.

It turns out that all these things are impossible — the set of constructible numbers is known not to include transcendentals (like $\sqrt{\pi}$) or things like cube roots.

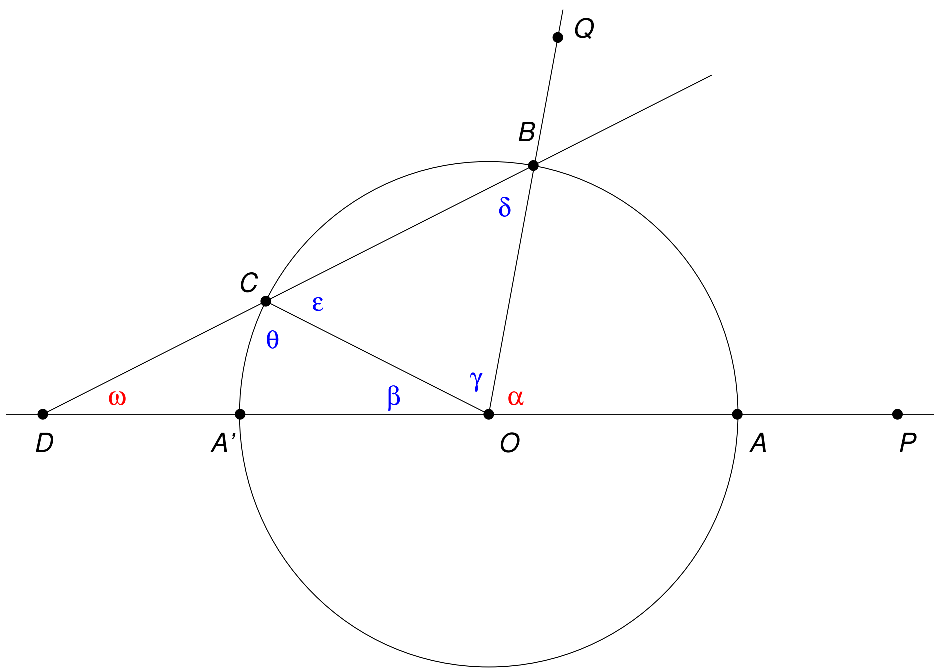
[1] Algebraically, this says exactly that the set $\mathbb{K}$ is a field.

3 How to trisect the angle

The following solution of the angle trisection problem is attributed to Archimedes. While the construction works, it is not Euclidean because it uses a new tool: a straightedge with two points marked on it.

Let $\angle POQ$ be the angle to be trisected. Draw a circle Z centered at O, and call the radius of the circle r. Let A and A' be the points where $\overleftrightarrow{OP}$ meets Z (with A between O and P) and let B be the point where $\overrightarrow{OQ}$ meets Z.

Now, here's the non-Euclidean step. Mark two points on your straightedge at distance r. Then, move the straightedge so that the two marked points lie on $\overleftrightarrow{OP}$ and Z — say at points C and D — and so that the line of the straightedge goes through Q. Label the angles $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \omega$ as shown.



Remember that α is the angle we want to trisect. As we will see, $\omega = \alpha/3$. Here's why. By construction, we know that

$$OA = OB = OC = CD = r.$$

In particular, ΔOBC and ΔCOD are isosceles triangles, with vertices O and C respectively. Since the base angles of an isosceles triangle are equal, we infer that

$$\delta = \varepsilon \qquad \text{and} \tag{1a}$$
$$\beta = \omega. \tag{1b}$$

Next, we apply the theorem that the angles of a triangle add up to 180°. In particular,

$$\gamma + \delta + \varepsilon = 180^\circ \qquad \text{and} \tag{2a}$$
$$\beta + \theta + \omega = 180^\circ. \tag{2b}$$

Third, because A, O, A' are collinear and B, C, D are collinear, we know that

$$\alpha + \beta + \gamma = 180^\circ \qquad \text{and} \tag{3a}$$
$$\varepsilon + \theta = 180^\circ. \tag{3b}$$

Now, putting these six equations together and using some algebra, it is possible to show that $\omega = \alpha/3$. (The details are left to the reader.)