Notes on transformational geometry

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Contents

1 The intuition **1**

2 Basic definitions **2**

2.1 Transformations 2

2.2 Groups 4

2.3 Notation for transformations 5

2.4 Transformations and geometry 5

3 Special kinds of transformations: isometries, similarities, and affine maps **7**

4 The structure of isometries **8**

5 Symmetries of bounded figures **10**

5.1 An example 10

5.2 Defining figures by their symmetry groups 11

5.3 Regular polygons 12

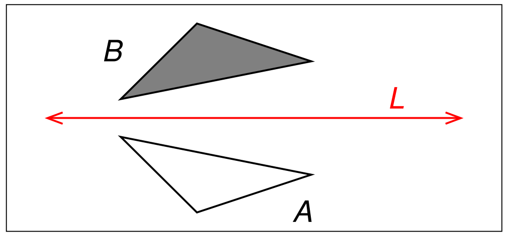
5.4 Other polygons 13

6 Counting symmetries **14**

1 The intuition

When we talk about transformations like reflection or rotation informally, we think of moving an object in unmoving space. For example, in the following diagram, when we say that the shaded triangle B is the reflection of the unshaded triangle A across the line L, we think about physically picking up A the unshaded triangle and reflecting it about the line.

Figure 1: Reflecting a triangle across a line



This is not how mathematicians think of transformations. To a mathematician, it is space itself (2D or 3D or...) that is being transformed. The shapes just go along for the ride.

To understand how this works, let's focus on the following basic transformations of the plane: *translations* along a vector; *reflections* about a line; *rotations* by an angle about a point. To help us consider these as transformations of the plane itself, you've been given a transparency sheet. You'll keep a piece of paper fixed on your desk. You'll move the transparency. The transparency represents what happens when you move the entire plane. The paper that stays fixed tells you where you started from.

Project 1. Start by drawing **a dot** on your paper. Take a transparency sheet, put it over your paper, and trace the dot. What can you do to the transparency (i.e., plane) so that the dots will still coincide? I.e., which translations, reflections, and rotations leave the dot fixed?

Now draw **two dots** on the bottom sheet and trace them on the transparency. The dots should be two different colors, say red and blue. What can you do to the transparency so the dots still coincide, red on red, blue on blue? I.e., which translations, reflections, and rotations leave the two dots fixed? Which translations, reflections, and rotations put the blue dot on top of the red dot and the red dot on top of the blue dot?

Now try this with **three dots** (in three different colors, say red, blue and green) which are not collinear. Which translations, reflections, and rotations leave the three dots fixed? What about **three dots of the same color?** What about **two red dots and one blue dot?**

Now try this with a **straight line.** (Of course you can't draw an infinitely long line on the paper, but you can draw a line segment and pretend.) Which translations, reflections, and rotations leave the line fixed? Which translations, reflections, and rotations don't leave the line fixed but still leave it lying on top of itself?

The idea of transformational geometry is that by studying the behavior of individual transformations, and how different transformations interact with each other, we can understand the objects being transformed.

2 Basic definitions

2.1 Transformations

Let's formally define what a transformation is:

Definition 1. A *transformation* of a space S is a map ϕ from S to itself which is 1-1 and onto. (Notation: $\phi : S \to S$.)

Notes on terminology:

- "Map" is just a synonym for "function". (It's a shorter word and sounds more geometric.)

- Remember that "1-1" means that if p, q are different points, then $\phi(p) \neq \phi(q)$ (that is, there's no more than one way to get to any given point in S via ϕ) while "onto" means that for every point q, there is some point p such that $\phi(p) = q$ (that is, there's at least one way to get to any given point via ϕ).
- A function that is both 1-1 and onto is also called a *bijection*.
- We'll often use Greek letters (like ϕ) for the names of transformations, and regular letters (like x) for the names of points and other sets.

Here are some examples of transformations of $\mathbb{R}^2$ (the plane):

1. Reflecting the plane across a line.
2. Rotating the plane about a point by a given angle.
3. Translating a plane by a given vector.
4. Contracting or expanding the plane about a point by a constant factor.
5. Doing absolutely nothing (i.e., sending every point to itself). This is called the *identity transformation.* It might not look very exciting, but it's an extremely important transformation, and it's certainly 1-1 and onto.

All of these kinds of transformations can be applied to $\mathbb{R}^3$ (3-space) as well, with some modification. For example, reflection in $\mathbb{R}^3$ takes place across a plane, not across a line, and rotation occurs around a line, not a point. (Question for those who have had some linear algebra or vector calculus: How do these various transformations behave in $\mathbb{R}^n$?)

Here are some functions that are *not* transformations:

1. The function taking all points $(x, y) \in \mathbb{R}^2$ to the point $x \in \mathbb{R}$. It's not 1-1, and the space you start with isn't the space you end up with.[1]
2. The map taking all points $x \in \mathbb{R}$ to the point $(x, 0) \in \mathbb{R}^2$. It's not onto, and the space you start with isn't the space you end up with (even though $\mathbb{R}$ is geometrically isomorphic to its image).
3. Folding a plane across a line L: this is 2-1 rather than 1-1 off L, and it isn't onto the whole plane.
4. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $\alpha(x) = x^2$. It's neither 1-1 nor onto. (On the other hand, the function $\beta(x) = x^3$ *is* a transformation.)

An important note. When we talk about transformations, we only care about where points end up, not how they get there. For example, the following three "recipes" all describe the same transformation:

- rotate the plane by $90°$ about the origin.
- rotate the plane by $-270°$ about the origin.
- reflect the plane across the x-axis, then reflect across the line $y = x$.

[1]This is still an interesting map geometrically, even though it isn't a transformation. It's an example of *projection*; in this case, projecting a plane onto a line.

To be precise: we consider two transformations $\phi : S \to S$ and $\psi : S \to S$ to be the same iff $\phi(p) = \psi(p)$ for all points p in S. It doesn't matter if ϕ and ψ are described by different recipes as long as they produce the same results.

Reflections, rotations and translations have a special property: they don't change the distance between any pair of points. That is, these transformations are *isometries*.[2] We'll come to back this idea later. For now, just notice that not every transformation is an isometry (for example, dilations are perfectly good transformations that are not isometries).

2.2 Groups

Transformational geometry has two aspects: it is the study of transformations of geometric space(s) and it studies geometry using transformations. The first thing people realized when they started to get interested in transformations in their own right (in the 19th century) was that there was an algebra associated with them. Because of this, the development of the study of transformations was closely bound up with the development of abstract algebra.

In particular, people realized that transformations behaved a lot like numbers in the following ways.

- *Closure.* Since transformations are 1-1 and onto functions, you can compose any two transformations to get another transformation. Specifically, if ϕ and ψ are transformations of a space S, then so is $\phi \circ \psi$. Remember, this means "first do ψ, then do ϕ", i.e.,

  $$\big(\phi \circ \psi\big)(p) = \phi(\psi(p)).$$

  It takes a little bit of checking to confirm that $\phi \circ \psi$ is 1-1 and onto (this is left as an exercise).

- *Existence of an inverse.* Recall the definition of the inverse of a function: $\phi^{-1}(p) = q$ if $\phi(q) = p$. For ϕ to have an inverse, it needs to be 1-1, but that's not a problem because it's part of the definition of a transformation. Also, inverting a function switches its domain and range, but in this case both domain and range are just S. So ϕ^{-1} is also a transformation of S.

- *Existence of an identity element.* The *identity transformation*, denoted "id", is the transformation that leaves everything alone: $\mathrm{id}(p) = p$ for all points $p \in S$. We've seen this before; it's certainly 1-1 and onto, so it's a transformation.

- *Associativity.* If ϕ, ψ and ω are three transformations of a space, then $\phi \circ (\psi \circ \omega) = (\phi \circ \psi) \circ \omega$. Indeed, for any point $x \in S$,

  $$\big(\phi \circ (\psi \circ \omega)\big)(x) = \phi(\psi(\omega(x))) = \big((\phi \circ \psi) \circ \omega\big)(x).$$

These four properties show up together in a lot of places. For instance, consider the set $\mathbb{R}$ of real numbers and the operation of addition. If you add two real numbers, you get a real number. Every real number has an additive inverse, namely its negative. There's an additive identity, namely 0. And addition is associative: $(a+b)+c = a+(b+c)$. (One way to think about associativity is that it doesn't matter how you parenthesize an expression like $a + b + c$.)

Or if you've taken linear algebra, you know that every vector space has these four properties.

Or consider the set of *nonzero* real numbers and the operation of multiplication. Again, the operation is closed and associative. The number 1 is the identity element, and every real number r has the multiplicative inverse $1/r$.

[2]From Greek: "iso" = same, "metry" = distance.

These properties together — we can compose two transformations to get a new transformation; there is an identity transformation; every transformation has an inverse; and composition is associative — say that the transformations of a given space form an algebraic structure called a *group*. Analogously, the real numbers form a group because we can add two real numbers to get a real number; there is an additive identity; every real number has an additive inverse; and addition is associative.

One big difference between the group of real numbers and the group of transformations is that addition is commutative, but composition of transformations is not. That is, if r, s are real numbers, then $r+s=s+r$, but if ϕ, ψ are transformations, then it is rarely the case that $\phi \circ \psi = \psi \circ \phi$. That's okay — the operation that makes a set into a group doesn't have to be commutative (but it does have to be associative).

The idea of a group is absolutely fundamental in mathematics.[3] As we'll see later on, groups come up all the time in geometry. In some sense, a lot of modern geometry is about groups just as much as it is about things like points and lines.

2.3 Notation for transformations

Here are the major types of transformations of the plane that we'll study:

Transformation	**Notation**
Reflection across line L	r_L
Rotation about point x by angle θ	$\rho_{x,\theta}$
Translation by vector $\vec{v}$	$\tau_{\vec{v}}$
"Glide reflection": first reflect across line L, then translate by vector $\vec{v}$	$\gamma_{L,\vec{v}}$
Dilation about point x with constant factor k	$\delta_{x,k}$

Most of these Greek letters are mnemonics for the type of transformation they denote (ρ = rho = rotation; τ = tau = translation; γ = gamma = glide reflection; δ = delta = dilation). The exception is r for reflection.

In some sense, these are the "most interesting" kinds of transformations (though certainly not all possible transformations).

This notation makes it easier to describe relations between transformations. For example, the fact that reflecting about a line twice ends up doing nothing can be expressed by the following equation: $r_L \circ r_L = \text{id}$. Instead of saying, "Rotating counterclockwise about a point x by angle θ is the inverse transformation of rotating clockwise about x by the same θ" — which is true, but extremely awkward — we can write the equation $(\rho_{x,\theta})^{-1} = \rho_{x,-\theta}$.

Observe that we are writing equations about transformations without reference to the points they are transforming. It is very convenient to be able to do this!

2.4 Transformations and geometry

In the previous section we looked at transformations by themselves. Now we look at the interaction between transformations and sets of points.

[3] To learn more about groups, take Math 558.

First, one piece of notation. If $\phi : S \to S$ is a transformation of S and A is a subset of S, then we'll write $\phi[A]$ for the image of A under S. That is,

$$\phi[A] = \{\phi(p) \mid p \in A\}.$$

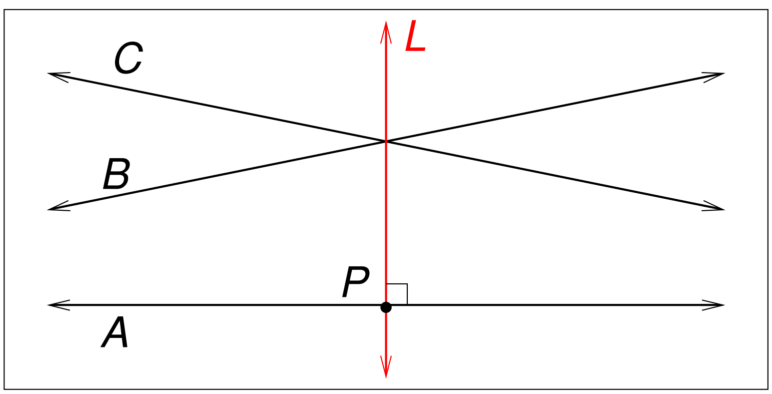
Symbol for symbol, this notation says: "$\phi[A]$ is the set of all points $\phi(p)$, where p is any point in A." For example, in Figure 1, where triangles A and B are each other's reflections across line L, we could write $r_L[A] = B$ and $r_L[B] = A$.

Definition 2. A transformation ϕ *fixes a point* p iff $\phi(p) = p$. It *fixes a set* A iff for all $p \in A$, $\phi(p) = p$. It is a *symmetry of* A iff $\phi[A] = A$.

Notice the big difference between ϕ **fixing** a set A (which means that every point in A is mapped to itself by ϕ) and being a **symmetry** of A (which just means that every point in A is mapped to some other point in A). So fixing a set is a much stronger condition than being a symmetry of it.

Every set has at least one symmetry — namely, the identity transformation, which fixes every point and therefore fixes every set.

Example 1. Consider the following picture.



What happens to lines A, B, C under the reflection r_L?

1. First of all, r_L fixes every point on L itself. So, certainly, $r_L[L] = L$.
2. Second, $r_L[A] = A$. On the other hand, r_L does *not* fix most of the points on A (except for P); it flips them across L to other points that are also on A. So r_L is a symmetry of A, but does not fix it.
3. Third, $r_L[B] = C$ and $r_L[C] = B$. So r_L is not a symmetry of B or of C.

Some more brief examples to think about:

1. If x is a point then $\rho_{x,\theta}$ fixes x, no matter what θ is.
2. If L is a line and $x \in L$ then $\rho_{x,180^\circ}$ is a symmetry of L, but does not fix it.
3. If L and M are perpendicular lines, then r_L is a symmetry of M, but does not leave it fixed. On the other hand, r_L does leave L itself fixed.
4. If $\vec{v} \neq 0$, then $\tau_{\vec{v}}$ does not have any fixed points. On the other hand, if L is parallel to $\vec{v}$, then $\tau_{\vec{v}}$ is a symmetry of L.

Using these terms, we can rephrase the questions asked in Project 1: Which transformations fix a single point? two points? three points? a line? Which transformations are symmetries of two points? of a line?

3 Special kinds of transformations: isometries, similarities, and affine maps

The next step is to categorize transformations according to how much geometric structure they preserve.

For example, consider the transformation of the plane that takes the point (x, y) to the point (x, y^3). Let's call this transformation ϕ. Note that ϕ is 1-1 and onto, so it is indeed a transformation. On the other hand, ϕ is not very nice from a geometric standpoint. For instance, ϕ takes the line $y = x$ and turns it into the curve $y = x^3$. So it doesn't preserve straight lines. And this means it doesn't preserve the angle $180°$, so it doesn't preserve angles. It doesn't preserve distances either: for example, the points $(1, 1)$ and $(1, 2)$ are at distance 1 from each other, but ϕ sends them to $(1, 1)$ and $(1, 8)$, which are at distance 7. ϕ is an example of the kind of transformation we are not interested in.

Definition 3. Suppose we have a transformation $\phi : S \to S$.

1. ϕ is an *isometry* iff it preserves distances. That is, if X and Y are any two points, then $XY = X'Y'$, where $X' = \phi(X)$ and $Y' = \phi(Y)$.

2. ϕ is a *similarity* iff it preserves angles: that is, if X, Y, Z are any three points, then $\angle XYZ \cong \angle X'Y'Z'$, where $\phi(X) = X'$, $\phi(Y) = Y'$ and $\phi(Z) = Z'$.

3. ϕ is an *affine map* iff it preserves straight lines: that is, if A is a line, then so is $\phi[A]$, and if $\phi[A]$ is a line, then so is A.

We've already observed that if ϕ is any rotation, reflection, or translation, then it is an isometry. Therefore, ϕ is also a similarity and an affine map. Dilations are similarities and are affine maps, but not isometries.

In fact, the ideas of isometry, similarity, and affine map are successively more and more general:

Theorem 1. *(1) Every isometry is a similarity, but not every similarity is an isometry.*
(2) Every similarity is affine, but not every affine map is a similarity.

Proof. (1) Suppose ϕ is an isometry. Let X, Y, Z be three points and let $\phi(X) = X'$, $\phi(Y) = Y'$ and $\phi(Z) = Z'$. We want to prove that $\angle XYZ \cong \angle X'Y'Z'$. By definition of isometry, we know that $XY = X'Y'$, $XZ = X'Z'$, and $YZ = Y'Z'$. But then $\Delta XYZ \cong \Delta X'Y'Z'$ by SSS. Therefore $\angle XYZ \cong \angle X'Y'Z'$, so we have proved that ϕ is a similarity.

On the other hand, dilations are similarities, but not isometries.

(2) If ϕ is a similarity, then it preserves angles, so in particular it preserves the angle $180°$. That is, it preserves straight lines. (More precisely, if three points are collinear, then so are their images under ϕ, and if three points are not collinear, then neither are their images.)

To finish the proof, we need to come up with a transformation that is an affine map, but not a similarity — this is left as an exercise. (Note: By part (1) of the proof, the desired transformation cannot be an isometry, since then it would be a similarity as well.) ☐

Theorem 2. *The isometries form a group, the similarities form a larger group, and the affine maps form a still larger group.*

Proof. We'll just consider the case of isometries — the proofs that the other two sets are groups work exactly the same way. To prove that the set of isometries forms a group, we show that it satisfies the four conditions listed in Section 2.2.

1. *Closure.* We need to show that the composition of two isometries is an isometry, i.e., that if ϕ and ψ preserve distance, then so does $\psi \circ \phi$. Let X, Y be any two points and let $X' = \phi(X)$, $Y' = \phi(Y)$, $X'' = \psi(X')$, $Y'' = \psi(Y')$. Then $XY = X'Y'$ (because ϕ is an isometry) and $X'Y' = X''Y''$ (because ψ is an isometry), but that means that $XY = X''Y''$, and $X'' = (\psi \circ \phi)(X)$ and $Y'' = (\psi \circ \phi)(Y)$. Therefore, $\psi \circ \phi$ is an isometry by definition.

2. *Inverses.* Suppose that ϕ is an isometry. In particular ϕ is a transformation, so it has an inverse transformation ϕ^{-1}, which we want to show is affine. So, let X, Y be any two points and let $X^* = \phi^{-1}(X)$, $Y^* = \phi^{-1}(Y)$. Then $\phi(X^*) = X$ and $\phi(Y^*) = Y$. Since ϕ is an isometry, $X^*Y^* = XY$. That's exactly what we need to show that ϕ^{-1} is an isometry.

(These were the hard parts.)

3. *Identity element.* The identity transformation is an isometry, because clearly $XY = \mathrm{id}(X)\,\mathrm{id}(Y)$.

4. *Associativity.* Isometries are functions, so their composition satisfies the associative law. □

4 The structure of isometries

In this section we focus on isometries. There are three major theorems about isometries. Two of their proofs are fairly complicated, so we won't give them. But we will give applications.

Theorem 3 (The Three-Point Theorem). *Every isometry of the plane is determined by what it does to any three non-collinear points. That is, if ϕ, ψ are isometries and A, B, C are non-collinear points such that $\phi(A) = \psi(A)$, $\phi(B) = \psi(B)$, and $\phi(C) = \psi(C)$, then $\phi = \psi$.*

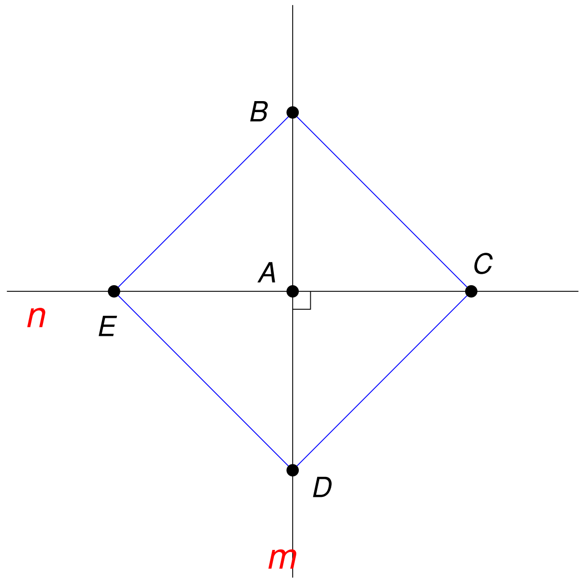
The proof of this theorem is rather technical, but you've already seen the idea behind it — think about the three-dot example in Project 1.

The Three-Point Theorem is useful for checking whether two isometries are equal: all you have to do is check that they agree on each of three non-collinear points. (Of course, you may have to use some ingenuity in choosing those points appropriately.)

Example 2. Suppose that m, n are perpendicular lines that meet at a point A (see figure below). We will prove that

$$r_m \circ r_n = \rho_{A,180^\circ}.$$

We need to find three noncollinear points and describe what each of these two isometries — the composition of reflections $r_m \circ r_n$, and the rotation $\rho_{A,180^\circ}$ — does to them. The point A is a clear choice for one of the three points. For the others, let's draw a square $BCDE$ centered at A with its diagonals parallel to m and n (shown in blue below). (Why? Because all the transformations we've described are symmetries of this square, so it's easy to see what they do to its vertices.)



We see that

$$
\begin{aligned}
r_n(A) &= A, & r_n(B) &= D, & r_n(C) &= C,\\
r_m(A) &= A, & r_m(D) &= D, & r_m(C) &= E,
\end{aligned}
$$

and therefore

$$
(r_m \circ r_n)(A) = A, \qquad (r_m \circ r_n)(B) = D, \qquad (r_m \circ r_n)(C) = E.
$$

On the other hand,

$$
\rho_{A,180^\circ}(A) = A, \qquad \rho_{A,180^\circ}(B) = D, \qquad \rho_{A,180^\circ}(C) = E.
$$

So the three non-collinear points A, B, C are mapped to the same points—namely A, D, E respectively—by $r_m \circ r_n$ and $\rho_{A,180^\circ}$. Therefore, by the Three-Point Theorem, $r_m \circ r_n = \rho_{A,180^\circ}$, which is what we were trying to prove. □

Theorem 4 (**The Three-Reflection Theorem**). *Every isometry is the composition of at most three reflections.*

If you believe the Three-Point Theorem, then you can prove the Three-Reflection Theorem constructively (and in fact you will do so as a homework problem). That is, if ψ is any isometry, then the Three-Point Theorem says that ψ is defined by what it does to any three non-collinear points A, B, C. So, to prove the Three-Reflection Theorem, it is sufficient to show that if A, B, C, A^*, B^*, C^* are six points such that $\Delta ABC \cong \Delta A^*B^*C^*$, then there is some way of transforming ΔABC to $\Delta A^*B^*C^*$ using three or fewer reflections.

One application of the Three-Reflection Theorem is the following theorem — which, in case you thought everything was about the number 3, is about the number 4.

Theorem 5 (**The Isometry Classification Theorem**). *Every isometry is either a reflection, a rotation, a translation, or a glide reflection.*

(What about the identity? It can be described as either translation by the zero vector, or as rotation about any point by 0°.)

There are many ways to prove this theorem, all of them tedious, so we won't give a proof. But all of the proofs rely to some extent on the Three-Reflection Theorem. On the other hand, the Isometry Classification Theorem has a nice corollary.

Theorem 6. *Let ϕ be an isometry. Either ϕ is a symmetry of some line or it fixes a point.*

Proof. By the Isometry Classification Theorem, there are only four cases to consider.

- If ϕ is a reflection r_ℓ, then ϕ fixes every point on ℓ, so it certainly is a symmetry of ℓ.
- If ϕ is a rotation $\rho_{p,\theta}$, then it fixes the point p.
- If ϕ is a translation $\tau_{\vec{c}}$, then it is a symmetry of any line parallel to the translation vector $\vec{v}$.
- If ϕ is a glide reflection $\gamma_{\ell;v}$, then it is a symmetry of the line ℓ. □

How many symmetries does a regular tetrahedron have?

How about a cube? Or an icosahedron?

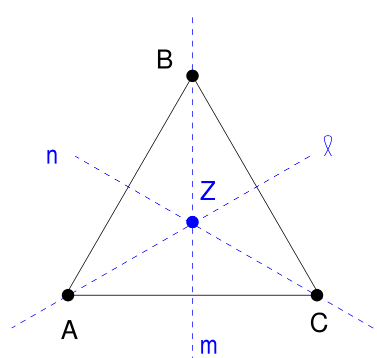
5 Symmetries of bounded figures

What can we say about the set of symmetries of a figure?[4] First of all, let's agree that when we talk about a symmetry of a figure, we restrict ourselves to isometries. This captures our intuition. There are many transformations that look like isometries in a small region of space but then do strange things outside it, and it complicates our discussion too much to talk about those.

Call the figure $\mathscr{F}$. The symmetries of $\mathscr{F}$ are closed under composition; the identity transformation of the plane is a symmetry of $\mathscr{F}$; and each symmetry of $\mathscr{F}$ has an inverse which is also a symmetry of $\mathscr{F}$. So they form a group, which we'll call $\mathrm{Sym}(\mathscr{F})$.

5.1 An example

Suppose that $\mathscr{F}$ is an equilateral triangle ΔABC. In addition to the identity, the group $\mathrm{Sym}(\Delta ABC)$ contains two notrivial rotations — namely $\rho_{Z,120}$ and $\rho_{Z,240}$, where Z is the center of the triangle — and three reflections: r_ℓ, r_m, and r_n, where ℓ, m, n are the bisectors of the three sides of the triangle.



[4]Here I'm using the Euclidean definition of "figure": an object built out of curves and line segments. So triangles, pentagons and circles are figures, but not, for example, a filled-in circle.

That is,

$$\mathrm{Sym}(\Delta ABC) \;=\; \Big\{\mathrm{id},\ \rho_{Z,120},\ \rho_{Z,240},\ r_\ell,\ r_m,\ r_n\Big\}.$$

Here's how we know that this is the complete list of symmetries. Every symmetry ϕ of ΔABC takes vertices to vertices; that is, ϕ is a symmetry of the set $\{A, B, C\}$. (For example, r_m fixes B and swaps A with C, while $\rho_{Z,120}$ maps A to C, B to A, and C to B. The identity, of course, fixes each of the three vertices.)

On the other hand, by the Three-Point Theorem, any isometry is determined by what it does to A, B and C. So there are only $3! = 6$ possibilities, which means that we've listed them all.

Since the symmetries form a group, we can ask how they behave under composition. That is, if ϕ, ψ are transformations in $\mathrm{Sym}(\Delta ABC)$, then which element of $\mathrm{Sym}(\Delta ABC)$ equals $\phi \circ \psi$? This question is really a set of thirty-six questions (e.g., What is $\rho_{Z,120} \circ r_m$? What is $r_n \circ r_n$?), whose answers can be collected in a table. The easiest way to calculate a single composition is to see what it does to A, B, C. For example,

$$\begin{aligned}
\rho_{Z,120}(r_\ell(A)) &= \rho_{Z,120}(A) = C = r_m(A); & \qquad r_\ell(r_m(A)) &= r_\ell(C) = B = \rho_{Z,240}(A),\\
\rho_{Z,120}(r_\ell(B)) &= \rho_{Z,120}(C) = B = r_m(B); & \qquad r_\ell(r_m(B)) &= r_\ell(B) = C = \rho_{Z,240}(B),\\
\rho_{Z,120}(r_\ell(C)) &= \rho_{Z,120}(B) = A = r_m(C); & \qquad r_\ell(r_m(C)) &= r_\ell(A) = A = \rho_{Z,240}(C);
\end{aligned}$$

so $\rho_{Z,120} \circ r_\ell = r_m$ and $r_\ell \circ r_m = \rho_{Z,240}$.

In the following table, the rows and columns are labeled by the elements of $\mathrm{Sym}(\Delta ABC)$, and the entry in column ϕ and row ψ is $\phi \circ \psi$.

	id	$\rho_{Z,120}$	$\rho_{Z,240}$	r_ℓ	r_m	r_n
id	id	$\rho_{Z,120}$	$\rho_{Z,240}$	r_ℓ	r_m	r_n
$\rho_{Z,120}$	$\rho_{Z,120}$	$\rho_{Z,240}$	id	r_m	r_n	r_ℓ
$\rho_{Z,240}$	$\rho_{Z,240}$	id	$\rho_{Z,120}$	r_n	r_ℓ	r_m
r_ℓ	r_ℓ	r_m	r_n	id	$\rho_{Z,120}$	$\rho_{Z,240}$
r_m	r_m	r_n	r_ℓ	$\rho_{Z,240}$	id	$\rho_{Z,120}$
r_n	r_n	r_ℓ	r_m	$\rho_{Z,120}$	$\rho_{Z,240}$	id

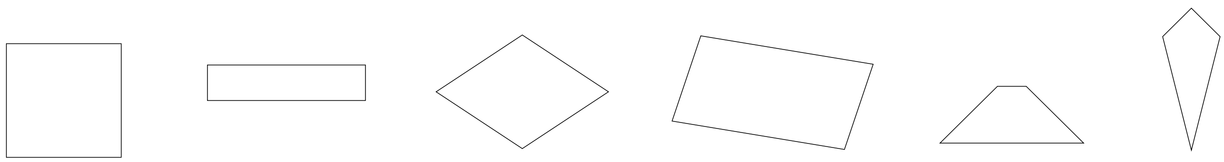
Notice the following things.

- It is not always true that $\phi \circ \psi =\neq \psi \circ \phi$. For example, if $r_\ell \circ r_m = \rho_{Z,240}$, but $r_m \circ r_\ell = \rho_{Z,120}$.
- The composition of two rotations, or of two reflections, is a rotation, while the composition of a rotation and a reflection (in either order) is a reflection. (This is analogous to the sign of the product of two real numbers: the product of two positive numbers or of two negative numbers is positive, while the product of a positive number with a negative number is negative.)

5.2 Defining figures by their symmetry groups

To a modern geometer (i.e., any geometer since the late 19th century), what characterizes a geometric figure isn't the number or characteristics of its sides and/or angles, but its symmetry group.

For example, suppose that $\mathscr{F}$ is a convex quadrilateral. Most of the time, $\mathscr{F}$ has only one symmetry, namely the identity transformation. But we can identify several special kinds of quadrilaterals.



What makes these special quadrilaterals special is exactly that they have a lot of symmetries. In fact, we can define them in terms of their symmetry groups.

A parallelogram is a convex quadrilateral whose only non-trivial[5] symmetry is $\rho_{C,180}$ for some point C (which we call the *center* of the parallelogram; it's where the diagonals meet).

A rectangle is a convex quadrilateral with one non-trivial rotational symmetry and two reflection symmetries. So is a rhombus.

An isosceles trapezoid has just the identity and one reflection symmetry.

A square, of course, has the most symmetries of any quadrilateral: four rotational symmetries (including the identity) and four reflection symmetries.

By the way, we could define a circle as follows. Let p be a point. A *circle with center* p is a figure $\mathscr{F}$ such that every rotation around p is a symmetry of $\mathscr{F}$, and every reflection across a line containing p is a symmetry of $\mathscr{F}$. This seems like a roundabout way to define a circle, but if you think about it, it's correct — every circle certainly has these symmetries, and any figure with these symmetries has to be a circle.

A figure is called *bounded* if it fits inside some circle. (For example, a triangle is bounded; a line isn't.) Not all isometries occur as symmetries of bounded figures.

Theorem 7. *If $\mathscr{F}$ is a bounded figure in the plane, then every symmetry of $\mathscr{F}$ is either the identity, a rotation, or a reflection. Equivalently, no translation or glide reflection can possibly be a symmetry of $\mathscr{F}$.*

The equivalence of these two statements comes from the Isometry Classification Theorem. Here's a really slick proof that depends on the definition of circle.

Proof. If $\mathscr{F}$ is bounded, then there is some unique smallest circle $\mathscr{C}$ that $\mathscr{F}$ fits inside. Every symmetry of $\mathscr{F}$ must be a symmetry of $\mathscr{C}$, and by the definition of circle, must be either a reflection or a rotation. □

In particular, this means that every symmetry of $\mathscr{F}$ fixes at least one point, namely the center c of $\mathscr{C}$. We could call this point the "center" of $\mathscr{F}$. For example, if $\mathscr{F}$ is a triangle then $\mathscr{C}$ is the circumscribed circle, so c is the circumcenter of $\mathscr{F}$ (that is, the intersection of the perpendicular bisectors of the sides).

5.3 Regular polygons

Most figures don't have any nontrivial symmetries. For example, if you draw a random triangle then it will almost certainly be scalene, and the only isometry that fixes it will be the identity. However, there are

[5] By "non-trivial", we mean "other than the identity".

special figures with more symmetries. The equilateral triangle of Section 5.1 is an example of this. More generally:

Definition 4. A polygon P is *regular* if all of its sides are congruent, and all of its angles are congruent.

Suppose p is a regular n-sided polygon (or "n-gon"). How big is the set $\mathrm{Sym}(P)$? First of all, let c be the center of P. If $\phi \in \mathrm{Sym}(P)$ then $\phi(c) = c$. Also, ϕ takes vertices to vertices, and it preserves adjacency among vertices; if v and w are vertices of P that are adjacent to each other, then so are $\phi(v)$ and $\phi(w)$. But the points c, v, w are noncollinear — but by the Three-Point Theorem, ϕ is determined by what it does to all of them. There are clearly n possibilities for $\phi(v)$ (namely, all the vertices of P) and once we know $\phi(v)$, there are 2 possibilities for $\phi(w)$ (namely, the vertices adjacent to $\phi(v)$), so we conclude that P has exactly $2n$ symmetries. In fact, it is not too hard to say what the symmetries are.

Theorem 8. *Let P be a regular polygon with n sides. The non-trivial symmetries of P are as follows:*

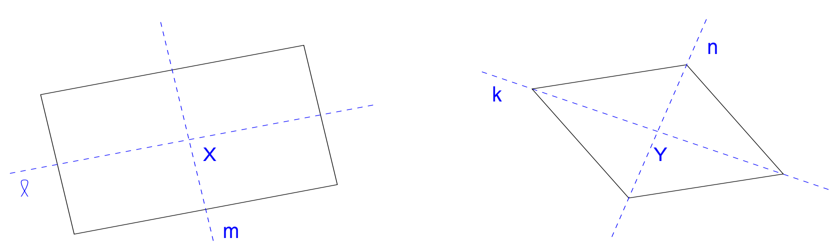
- *all reflections across its angle bisectors;*
- *all reflections across the perpendicular bisectors of its sides; and*
- *all rotations about its center by $(360k/n)^\circ$, for $0 < k < n$.*

Proof. All of these transformations are certainly symmetries of P. On the other hand, if $\phi \in \mathrm{Sym}(P)$, then $\phi(c) = c$, where c is the center of P (as defined above), so ϕ must either be a rotation about p or a reflection across a line containing c, and any rotation or not reflection that is not one of those listed above does not take vertices to vertices. ☐

5.4 Other polygons

What about polygons that are not regular, but have lots of symmetries nevertheless? For example, what does the group of symmetries of a rectangle look like?

Remember, we said that a rectangle is a convex quadrilateral with one non-trivial rotational symmetry and two reflection symmetries (across the perpendicular bisectors of each pair of opposite sides). Of course, so is a rhombus — although in this case the lines of reflection symmetry are the diagonals.



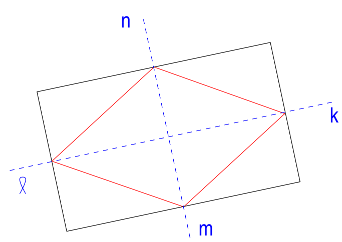
Here's the multiplication table for the symmetry group of a rectangle:

	id	$\rho_{X,180}$	r_ℓ	r_m
id	id	$\rho_{X,180}$	r_ℓ	r_m
$\rho_{X,180}$	$\rho_{X,180}$	id	r_m	r_ℓ
r_ℓ	r_ℓ	r_m	id	$\rho_{X,180}$
r_m	r_m	r_ℓ	$\rho_{X,180}$	id

And here's the multiplication table for the symmetry group of a rhombus:

	id	$\rho_{Y,180}$	r_k	r_n
id	id	$\rho_{Y,180}$	r_k	r_n
$\rho_{Y,180}$	$\rho_{Y,180}$	id	r_n	r_k
r_k	r_k	r_n	id	$\rho_{Y,180}$
r_n	r_n	r_k	$\rho_{Y,180}$	id

These two multiplication tables are essentially the same: if you take the first table and replace X with Y, ℓ with k, and m with n, you get the second table. Algebraically, we say that the symmetry groups of the rectangle and the rhombus are *isomorphic*. There's a good reason for this: the two figures can be superimposed so that their symmetry groups consist of exactly the same sets of transformations.

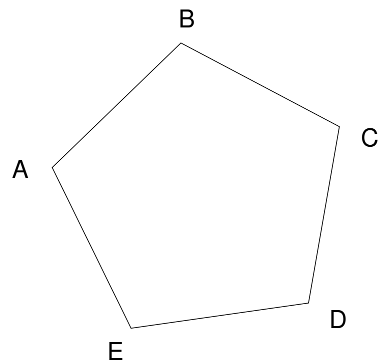


This is an example of how modern mathematics uses groups to study geometric objects. The fact that the symmetry groups of the rhombus and rectangle are the same indicates that there's some relationship between the two figures. Of course you don't need groups to realize that you can form a rhombus by joining the midpoints of a rectangle, but the same technique can be applied to more complicated figures.

6 Counting symmetries

If we know the symmetry group of an object (that is, if we know its multiplication table), then of course we know how many symmetries there are. But it is often possible to count the symmetries without having to work out the full symmetry group.

Example 3. Let $\mathcal{P} = ABCDE$ be a regular pentagon. Every symmetry ϕ of $\mathcal{P}$ permutes its five vertices (that is, it is a symmetry of the five-point set $\{A, B, C, D, E\}$), and, by the Three-Point Theorem, is completely determined by what it does to any three of the five.



But actually, ϕ is determined by even less information. For instance, if we know what $\phi(A)$ and $\phi(B)$ are, then we know ϕ completely (by the Three-Point Theorem again — because $\phi(O) = O$, where O is the center of $\mathcal{P}$, and the points A, B, O are non-collinear). The point $\phi(A)$ can be any of the 5 vertices of $\mathcal{P}$, and once we know $\phi(A)$, we know that $\phi(B)$ must be one of the 2 vertices sharing a side with $\phi(A)$ (whatever that is). Therefore, the number of symmetries of $\mathcal{P}$ is $5 \cdot 2 = 10$.

We can describe a symmetry of $\mathcal{P}$ by its *permutation word*. That is, write the five letters A, B, C, D, E in the order $\phi(A), \phi(B), \ldots, \phi(E)$. Here are the permutation words for all ten symmetries of the regular pentagon:

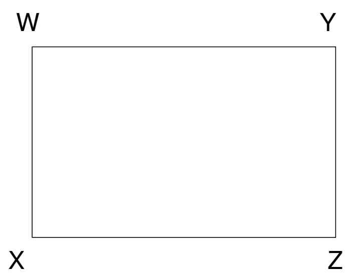
$$\begin{array}{ccccc} ABCDE, & AEDCB, & BAEDC, & BCDEA, & CBAED, \\ CDEAB, & DCBAE, & DEABC, & EABCD, & EDCBA. \end{array}$$

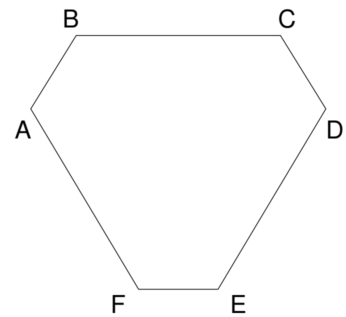
For example, $\rho_{O,144^\circ}$ correponds to the permutation word $DEABC$ (because $\rho_{O,144^\circ}(A) = D$, $\rho_{O,144^\circ}(B) = E$, etc.) and r_b corresponds to the permutation word $CBAED$ (because $r_b(C) = A$, $r_b(B) = B$, etc.) The permutation word $ABCDE$ corresponds to the identity transformation. Notice that each of the 5 possible first letters occurs twice in the table, once with each of its neighbors next to it. This corresponds exactly to our earlier observation.

Generalizing this argument, we can see that every regular polygon with n sides has exactly $2n$ symmetries. Of course, we already knew that from Theorem 8, but it's nice to confirm it another way. This method of counting symmetries doesn't tell us explicitly what the symmetries are, but on the other hand it is applicable to lots and lots of geometric objects — not just in the plane, but also in three-dimensional space (and even in four- and higher-dimensional spaces!)

Example 4. Let $\mathcal{R} = WYXZ$ be a rectangle that is not a square, as shown below, and let ϕ be a symmetry of $\mathcal{R}$. Then $\phi(X)$ can be any of the four vertices, but once we know $\phi(X)$, there's only one possibility for ϕ, because $\phi(W)$ must be the vertex adjacent to $\phi(X)$ by one of the short sides of $\mathscr{R}$. (If $\mathscr{R}$ were a square, then there would be two choices for $\phi(W)$ instead of one.) So $|\,\mathrm{Sym}(\mathcal{R})| = 4$. which confirms what we found earlier. The permutation words for the four symmetries are

$$WXYZ, \qquad XWZY, \qquad YZWX, \qquad ZYXW.$$

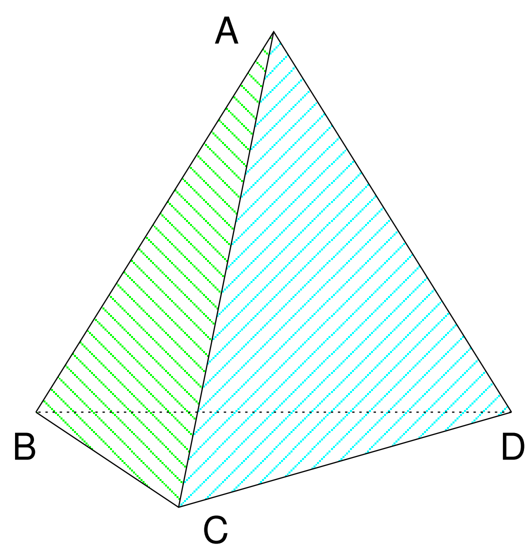




Similarly, let $\mathcal{H} = ABCDEF$ be the hexagon you studied in homework problem TG 15, and let ψ be a symmetry of $\mathcal{H}$. Again, $\psi(A)$ can be any of the six vertices, but once you choose $\psi(A)$, you immediately know what ψ does to the other five vertices of $\mathcal{H}$. Therefore, $|\operatorname{Sym}(\mathcal{R})| = 6$. (There's nothing special about A; we could just have well argued that ψ is determined by $\psi(E)$, which can be any of the six vertices.)

What about higher-dimensional objects?

Example 5. Let $\mathcal{T}$ be a regular tetrahedron (i.e., a triangular pyramid in which every side is an equilateral triangle). Call the vertices A, B, C, D.



How many symmetries does $\mathcal{T}$ have? Equivalently what are all the permutation words of symmetries of $\mathcal{T}$?

If ϕ is a symmetry, then clearly $\phi(A)$ can be any of $\phi(A)$, $\phi(B)$, $\phi(C)$ or $\phi(D)$. Four choices there.

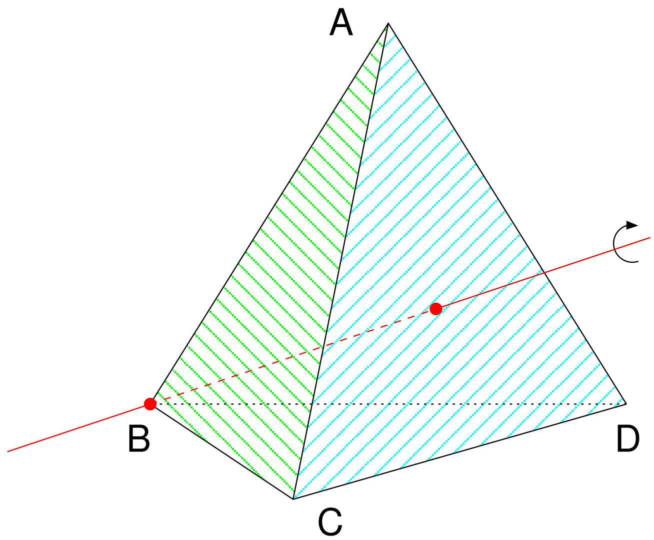
Having chosen $\phi(A)$, there are three choices for $\phi(B)$ (any of the other three vertices).

Having chosen $\phi(A)$ and $\phi(B)$, there are two choices for $\phi(C)$. Then, once we choose $\phi(C)$, there is only one possibility left for $\phi(D)$. of the other three vertices).

In total, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ symmetries of $\mathcal{T}$. In fact, *every* rearrangement of the letters A, B, C, D is a permutation word of a symmetry.

What these symmetries look like geometrically?

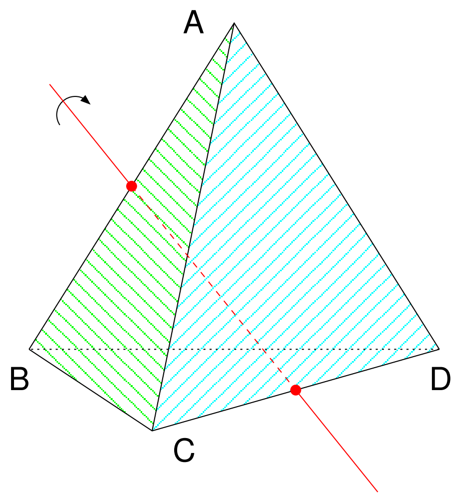
For example, you can draw a line connecting a vertex with the center of the opposite triangle and rotate $\mathcal{T}$ by 120° or 240° around this line, as in the following figure.

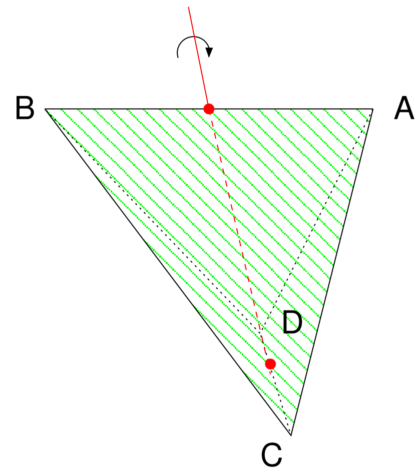


There are four ways to choose that vertex-triangle pair, so we get a total of eight rotations this way. Here are the permutation words.

Vertex	Opposite triangle	Permutation words
A	BCD	$ACDB,\ ADBC$
B	ACD	$CBDA,\ DBAC$
C	ABD	$BDCA,\ DACB$
D	ABC	$BCAD,\ CABD$

Another way to construct a rotation line is to connect the midpoints of two opposite edges of $\mathcal{T}$, as in the following figure. (It's probably easiest to visualize if you dangle $\mathcal{T}$ from one of its edges — the right-hand figure is an attempt at illustrating this.)





There are three such pairs of opposite edges: AB and CD; AC and BD; and AD and BC. This gives three more permutation words, respectively $BACD$, $CDAB$, and $DCBA$.

We've accounted for twelve symmetries so far (the identity and $3 + 8 = 11$ nontrivial rotations). The other twelve are reflections (for example, reflecting across the plane containing edge AC and the midpoint of edge BD) or compositions of reflections and rotations.