Math 243, Fall 2015
Extra Honors Problem for Week 9

In this multipart problem, you will come up with a conjecture for the volume of the ball B_r^n of radius r in $\mathbb{R}^n$ for *every* value of n and R. That is,

$$B_r^n = \{\mathbf{x} \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant r^2\}$$

and "volume" means "n-dimensional volume". Before you do anything else, review the calculation of the volume of the 4-dimensional ball from class on 10/26/15.

Part 1 [5 points]: Use hyperspherical coordinates in $\mathbb{R}^5$ to find the volume of B_r^5. If you wish, you may use a computer to calculate the Jacobian and to evaluate the resulting quintuple integral (quintegral?). You should, however, write down how the quintuple integral can be expressed as the product of five single integrals, using the multivariable extension of Problem #40(a) from §5.2 (one of last week's honors problems).

Part 2 [2 points]: Compare the volume elements dV for polar coordinates in $\mathbb{R}^2$, spherical coordinates in $\mathbb{R}^3$, and hyperspherical coordinates in $\mathbb{R}^4$ and $\mathbb{R}^5$. Conjecture a formula for the volume element in $\mathbb{R}^n$ in general. (In fact, this can be proven using induction and facts about determinants.)

Part 3 [2 points]: Conjecture a formula for the volume of B_r^n as the product of n integrals (or fewer).

Part 4 [3 points]: Check that your conjecture from Part 3 works for the cases you know, and make a table of values for enough additional values of n so that you are able to conjecture a closed-form formula (i.e., without integrals) for the volume of B_r^n. You will almost certainly want to use a computer; if so, include a printout of the relevant work you did with Matlab, Sage, WolframAlpha, etc. (Hint: Your conjecture should depend on the parity of n, i.e., whether n is odd or even.)