Math 243, Fall 2015
Extra Honors Problems for HW #11

Problem #1 [3 pts] You are trying to write an exam in a vector calculus class and would like to include a problem involving a scalar line integral. You come up with the idea of asking your students to find the surface area of a fence whose base is given by the curve $y = 1/x$, say from $(2, \frac{1}{2})$ to $(\frac{1}{2}, 2)$. You need to choose a height function $f(x, y)$ for the fence that results in an integral that is not trivial (for example, $f(x, y) = 0$ is not acceptable) but that can be evaluated with no more than Calculus I techniques (so nothing too exotic). Come up with such a function f.

Problem #2 [4 pts] Find the exact value of

$$\int_0^1 \frac{\cos(\pi t)\cos(3\pi t) + 3\sin(\pi t)\sin(3\pi t)}{\sin^2(\pi t) + \cos^2(3\pi t)}\, dt$$

without evaluating the integral numerically. (Hint: Interpret the integral topologically, using a calculator or computer to sketch an appropriate curve.)

You can use a calculator or computer to do the integral numerically and check your answer. However, to receive credit, you need to present an argument that does not at all rely on numerical integration.

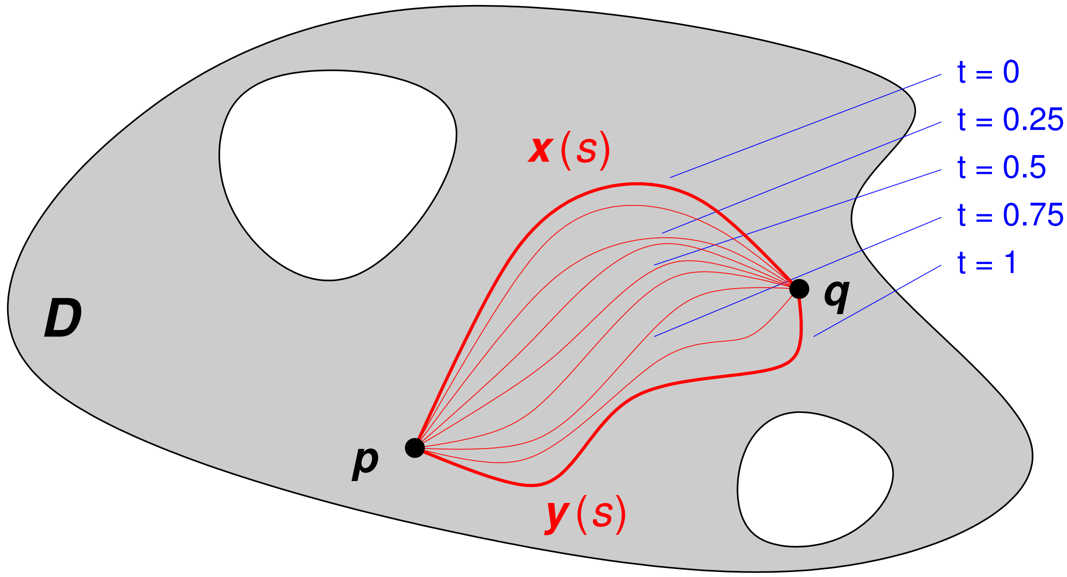
Problem #3 In this problem you will see a bit more about vector calculus can be used to investigate the topology of various spaces.

Let D be a region in $\mathbb{R}^2$, and let $\mathbf{x}(s)$ and $\mathbf{y}(s)$ be parameterized curves in D for $0 \leqslant s \leqslant 1$. (You can think of the parameter s as arclength if you want, but it doesn't necessarily have to be.) Assume also that $\mathbf{x}$ and $\mathbf{y}$ have the same starting and ending points: that is, $\mathbf{x}(0) = \mathbf{y}(0) = \mathbf{p}$ and $\mathbf{x}(1) = \mathbf{y}(1) = \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are points in D. We want to make precise the notion of "continuously changing $\mathbf{x}$ into $\mathbf{y}$." In other words, we want to hold $\mathbf{x}$ by its endpoints at $\mathbf{p}$ and $\mathbf{q}$ and let it wiggle around D. The endpoints have to stay steady, but the other points on the curve can move. So what we need is a 2-variable function

$$H : [0, 1] \times [0, 1] \to D$$

in which the first variable indicates position (s) and the second one indicates time (t). If we fix a value of t, say $t = a$, and let s vary, then we get a resulting parametrized curve $\mathbf{x}_a(s) = H(s, a)$ which you can think of as a "snapshot" of the curve at time $t = a$. (Note that every "snapshot" must stay inside the space D, just like $\mathbf{x}$ and $\mathbf{y}$.) As t increases from 0 to 1, that snapshot should evolve from $\mathbf{x}$ to $\mathbf{y}$. (See next page for a figure.) The following things have to be true:

- $H(s, 0) = \mathbf{x}(s)$ (since at time $t = 0$, the curve should look like $\mathbf{x}$).
- $H(s, 1) = \mathbf{y}(s)$ (since at time $t = 0$, the curve should look like $\mathbf{y}$).
- $H(0, t) = \mathbf{p}$ for all t (since the starting point does not change over time).
- $H(1, t) = \mathbf{q}$ for all t (since the ending point does not change over time).
- H is a continuous function (since we want the curve to evolve continuously).



A function like H is called a **homotopy between x and y**, and two curves that can be connected by a homotopy are called **homotopic**. Studying curves and their homotopies is an important way of understanding the topology of the space in which they live.

We know that if $\mathbf{x}$ and $\mathbf{y}$ are closed curves in D with the same starting and ending point, and $\mathbf{F}$ is a conservative vector field on D, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$. This is not necessarily true if $\mathbf{F}$ is merely irrotational. However, the following is true:

Theorem: If $\mathbf{F}$ is irrotational and $\mathbf{x}$ and $\mathbf{y}$ are homotopic, then $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$.

The proof of this fact relies essentially on Green's Theorem (plus some technical details) — you should think about why this is the case! (What happens when you let $\mathbf{x}$ evolve a little bit? How does that change $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$?)

(a) (5 pts): Suppose that $\mathbf{x}$ and $\mathbf{y}$ are closed curves in $\mathbb{R}^2$ with $\mathbf{x}(0) = \mathbf{x}(1) = \mathbf{y}(0) = \mathbf{y}(1) = \mathbf{0}$. Show that there is always a homotopy between them by constructing H explicitly in terms of $\mathbf{x}$ and $\mathbf{y}$. Hint: You can come up with an argument that works for every *convex* region $D \subseteq \mathbb{R}^2$ as well. ("Convex" means that if $\mathbf{p}$ and $\mathbf{q}$ are points in D, then the entire line segment from $\mathbf{p}$ to $\mathbf{q}$ is also contained in D.)

In fact, this conclusion is true for every simply-connected region (and can even be used as the definition of "simply-connected").

Part (b) (5 pts): Let $D = \mathbb{R}^2 - \{\mathbf{0}\}$. Show that if a, b are integers and $a \neq b$, then the two curves

$$\mathbf{x}(t) = (\cos 2\pi a t,\ \sin 2\pi a t), \qquad \mathbf{y}(t) = (\cos 2\pi b t,\ \sin 2\pi b t), \qquad 0 \leqslant t \leqslant 1$$

are **not** homotopic in D. (Hint: Use the Theorem, choosing the field $\mathbf{F}$ appropriately.)