Math 243, Fall 2015
Extra Honors Problem for HW #10

> Construct a C^∞ vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\operatorname{div} \mathbf{F}(x, y) \geqslant 0$ if $x \geqslant 0$ or $y \geqslant 0$, but $\operatorname{div} \mathbf{F}(x, y) < 0$ if $x < 0$ and $y < 0$. (I.e., the divergence is zero on the x- and y-axes, positive in the open first, second and fourth quadrants, and negative in the open third quadrant.)

The first step is going to be to find a scalar function f to use as $\nabla \cdot \mathbf{F}$. In other words, we want to find a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$ that is negative in the open third quadrant (i.e., when x and y are both negative), but nonnegative everywhere else. Only after we have found f will we think about reconstructing $\mathbf{F}$ from f.

First, let's consider a related, but simpler, problem in $\mathbb{R}$ instead of $\mathbb{R}^2$. Can we find a C^∞ function $h : \mathbb{R} \to \mathbb{R}$ that is negative when $x < 0$, but nonnegative when $x \geqslant 0$?

Most functions you know do not behave this way — the only "standard" functions that are constant on some interval are in fact constant of all of $\mathbb{R}$. However, we can take advantage of the fact that exponential functions decay incredibly quickly. As x approaches 0, the value of $-1/x$ tends to $-\infty$, and so the values of $e^{-1/x}$ approach 0. They do so fast. Very, very fast. How fast? So fast that all its higher derivatives approach zero — which means that the piecewise function

$$h(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

is of class C^∞. This fact needs to be proved, so...

Part 1: Prove that h is of class C^∞. In order to do so, you will have to show that all higher derivatives $h^{[k]}(x) = d^k h/dx^k$ are continuous at 0, i.e., that

$$\lim_{x \to 0} h^{[k]}(0) = 0$$

for every integer $k \geqslant 0$. Technically, the tool you need for this is mathematical induction, but if you don't know what that is, don't worry; check that this is true for $k = 0, 1, 2$ and make a reasonable argument that it will continue to be true. (If you feel like tackling the induction proof, it may be easier to prove the more general statement that $\lim_{x \to 0} d^k/dx^k \big(e^{-1/x} r(x)\big) = 0$ for *every* rational function $r(x)$.)

Having proven (1), what's next? (*If you got stuck on (1), don't panic! Just assume that $h(x)$ is C^∞ and keep going.*) The next step is to use $h(x)$ to constructing the function f we were looking for in the first place.

Part 2: Confirm that $g(x) = h'(x)$ is also of class C^∞ and is positive for all $x > 0$.

Why work with g instead of h? Because it can be antidifferentiated, which will be helpful when trying to construct a vector field $\mathbf{F}$ with specified divergence.

Part 3: Use g to construct a scalar function $f(x, y)$ of class C^∞ that is negative when x, y are both negative, and nonnegative otherwise. (Hint: Work one quadrant at a time.)

Part 4: Having done all this work, you should now be able to solve the original problem. Construct a vector field $\mathbf{F}$ whose divergence is f.

Part 5: Suppose we tighten up the problem constraints a bit. Can you construct a C^∞ vector field $\mathbf{G}(x, y)$ that is negative in the open quadrant ($x < 0$ and $y < 0$), zero on the negative x- and y-axes, and *strictly positive* everywhere else (i.e., if $x > 0$ or $y > 0$)? (Hint: Add another couple of pieces to the construction of $f(x, y)$.)