Math 223 Test #1 Solutions **Prof. Jeremy Martin**

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Problem #1 **(#1a)** Call the points $A = (1,1,1)$, $B = (1,2,3)$, $C = (2,4,5)$. The vectors $\mathbf{u} = \vec{AB} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = \vec{BC} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ lie in P. Therefore, their cross product

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

is a normal vector to P. You can confirm this by checking that $\mathbf{n} \cdot \mathbf{u} = 0$ and $\mathbf{n} \cdot \mathbf{v} = 0$.

Common mistake: Confusing the points with the direction vectors. The vector $(1,1,1) \times (1,2,3)$ (for example) is *not* normal to P.

(#1b) The vectors $\mathbf{u}$ and $\mathbf{v}$ above, together with a point in the plane), are all the information we need. Taking A as the base point gives the parametrization

$$(x,y,z) = A + s\mathbf{u} + t\mathbf{v} = (1,1,1) + s(0,1,2) + t(1,2,2) = (1+t, 1+s+2t, 1+2s+2t)$$

(there are many other possible parametrizations).

Common mistake: Using the normal vector instead of two vectors in the plane, thus coming up with the parametrization $(x,y,z) = (1,1,1) + t(-2,2,-1) = (1-2t, 1+2t, 1-t)$. This is actually the parametrization of the normal line through P at $(1,1,1)$, not of P itself. The number of parameters should equal the dimension of the space you are parametrizing; here P is 2-dimensional so there need to be two parameters.

Problem #2 **(#2a)** Convert to polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \;=\; \lim_{\substack{r\to 0\\ \theta\to ??}} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \;=\; \lim_{\substack{r\to 0\\ \theta\to ??}} r(\cos^3\theta + \sin^3\theta) \;=\; 0$$

because $\lim_{r\to 0} r = 0$ and $\cos^3\theta + \sin^3\theta$ is bounded (it's certainly in the range $[-2,2]$).

Common mistake: Plugging in $x = 0$, calculating the limit in y, getting 0, doing the same with $x = 0$, and then incorrectly concluding that the two-variable limit is 0. You have to ensure that the expression approaches 0 *along any path to* $(0,0)$.

(#2b) If $(x,y) \to (0,0)$ along the x-axis (i.e., $y = 0$) then the expression approaches

$$\lim_{x\to 0} \frac{x^3+0^3}{x^2+0^3} \;=\; \lim_{x\to 0} \frac{x^3}{x^2} \;=\; \lim_{x\to 0} x \;=\; 0.$$

If $(x,y) \to (0,0)$ along the x-axis (i.e., $x = 0$) then the expression approaches

$$\lim_{y\to 0} \frac{0+y^3}{0+y^3} \;=\; \lim_{x\to 0} 1 \;=\; 1.$$

Since these two paths do not agree, the original limit does not exist.

Common mistake: Converting to polar coordinates (so far, so good) and asserting that

$$\lim_{\substack{r\to 0\\ \theta\to ??}} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^3\sin^3\theta} = \lim_{\substack{r\to 0\\ \theta\to ??}} \frac{r\cos^3\theta + r\sin^3\theta}{\cos^2\theta + r\sin^3\theta} \underset{\text{wrong!}}{=} \frac{0+0}{\cos^2\theta + 0} \underset{\text{wrong!}}{=} 0.$$

The last two equalities are only valid if $\cos\theta \neq 0$, but not if $\cos\theta = 0$ (i.e., $\theta = \pi/2$, so (x, y) is approaching the origin along the y-axis).

Problem #3 **(#3a)** $\nabla k(x,y,z) = \left(\frac{\partial k}{\partial x}, \frac{\partial k}{\partial y}, \frac{\partial k}{\partial z}\right) = \left(e^y,\ xe^y + z/y,\ \ln y + 3z^2\right)$

(#3b) $\nabla k(0,1,2) = (e, 2, 12)$ is a normal vector to the level surface, so the tangent plane to the level surface has the equation $\nabla k(0,1,2)\cdot(\mathbf{x}-\mathbf{a}) = 0$, i.e.,

$$e(x-0) + 2(y-1) + 12(z-2) = 0 \quad \text{or} \quad ex + 2y + 12z = 26.$$

Note that the 666 is a red herring; the level surfaces of k would be the same regardless of what the constant is, and the equation for the tangent plane should be satisfied by the point $(x, y, z) = (0, 1, 2)$ itself.

Common mistake: Forgetting to plug $\mathbf{x} = \mathbf{a}$ into the gradient and coming up with the equation $\nabla k(x, y, z)\cdot(\mathbf{x}-\mathbf{a}) = 0$, i.e., $e^y x + (xe^y + z/y)(y-1) + (\ln y + 3z^2)(z-2)$; whatever the surface is defined by this equation, it is certainly not a plane.

(#3c) The answer is the same as (c): the gradient gives the direction of fastest increase. If you want to normalize, you can: $\|\nabla k(0,1,2)\| = \|(e,2,12)\| = \sqrt{e^2+148}$, so a unit vector in the direction of fastest increase is

$$\frac{1}{\sqrt{e^2+148}}(e\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}).$$

(#3d) Prof. Nitram's direction vector is $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ $(= (2,-2,8) - (0,1,2))$. Note that $\|v\| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$. A unit vector in the same direction is $\mathbf{u} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$, and so the answer to the problem is the directional derivative

$$D_{\mathbf{u}}k(\mathbf{a}) = \nabla k(\mathbf{a})\cdot\mathbf{u} = (e,2,12)\cdot\left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right) = \frac{2e-6+72}{7} = \frac{2e-66}{7}.$$

This is approximately -8.6519 (in degrees Kelvin per whatever the unit of distance is), but you should leave your answers in exact form.

Common mistakes:

- Forgetting to convert $\mathbf{v}$ to the unit vector $\mathbf{u}$. This results in an answer that is too large by a factor of $\|v\| = 7$.
- Using $(2,-2,8)$ as the direction vector (normalized or not).
- Plugging $(2,-2,8)$ into k or ∇k. Note that $k(2,-2,8)$ is undefined since it involves $\ln(-3)$.

Problem #4 Agh! The function I gave you didn't match the graph - there was a missing square root sign. The graph shown on the paper and displayed on the screen was actually of the function

$$g(x,y) = \begin{cases} \dfrac{|xy|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Accordingly, I gave credit for solutions that used either f or g.

(#4a) Both f and g are continuous on all points of $\mathbb{R}^2$ except $(0,0)$, since on those domains they are built out of absolute values, rational functions, and square roots (of positive numbers), all of which are continuous. At $(0,0)$, however, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{r\to 0\\ \theta\to ?}} \frac{|r^2\cos\theta\sin\theta|}{r^2} = \lim_{\substack{r\to 0\\ \theta\to ?}} |\cos\theta\sin\theta| \ \textbf{DNE},$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{\substack{r\to 0\\ \theta\to ?}} \frac{|r^2\cos\theta\sin\theta|}{|r|} = \lim_{\substack{r\to 0\\ \theta\to ?}} |r\cos\theta\sin\theta| = 0 = g(0,0),$$

so f is not continuous at $(0,0)$, but g is.

(#4b) This answer is the same for both f and g. Both $f_x(x,y)$ and $g_x(x,y)$ exist whenever x and y are both nonzero. They also both exist when $y=0$, because plugging in $y=0$ gives the partial function $g(x,0)=0$, which is differentiable. (I.e., the cross section of the graph by the plane $y=0$ is a horizontal line). However, if $x=0$ and $y\neq 0$, then

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{\frac{|hy|}{h^2+y^2}-0}{h} = \lim_{h\to 0} \frac{|hy|}{h(h^2+y^2)}$$

$$= \lim_{h\to 0} \frac{\operatorname{sign}(h)|y|}{(h^2+y^2)} = \lim_{h\to 0} \frac{\operatorname{sign}(h)|y|}{|y^2|} = \lim_{h\to 0} \frac{\operatorname{sign}(h)}{|y|} \ \textbf{DNE}$$

and

$$g_x(0,y) = \lim_{h\to 0} \frac{g(0+h,y)-g(0,y)}{h} = \lim_{h\to 0} \frac{\frac{|hy|}{\sqrt{h^2+y^2}}-0}{h} = \lim_{h\to 0} \frac{|hy|}{h\sqrt{h^2+y^2}}$$

$$= \lim_{h\to 0} \frac{\operatorname{sign}(h)|y|}{\sqrt{h^2+y^2}} = \lim_{h\to 0} \frac{\operatorname{sign}(h)|y|}{|y|} = \lim_{h\to 0} \operatorname{sign}(h) \ \textbf{DNE}.$$

You can also figure this out by looking at the graph — slicing with any plane $y=c$, where c is a nonzero scalar, cuts through the ridge at the bottom of the surface and gives a sharp point (i.e., a non-differentiable point) at $(0,c)$.

In summary, f_x exists if $x=0$ or $y\neq 0$ (or both), and likewise for g_x.

(#4c) If x,y are both nonzero, the function is differentiable. If $x\neq 0$ and $y=0$, then part (b) says that f_x does not exist, so f is not differentiable; likewise, if $x=0$ and $y\neq 0$, then f is not differentiable at (x,y) because f_y does not exist. At $(0,0)$, the partial derivatives exist <u>but are not continuous</u>, so f is not differentiable there either. Therefore, the open set on which f is differentiable is $\{(x,y)\in\mathbb{R}^2 \mid x\neq 0 \text{ and } y\neq 0\}$.

Problem #5 The Chain Rule says that $Dh(x,y) = [Dq(p(x,y))][Dp(x,y)]$. So,

$$Dp(x,y) = \begin{bmatrix} 2x & -1 \\ 1 & 2y \end{bmatrix}, \qquad Dq(s,t) = \begin{bmatrix} 2s & 0 \\ 1 & 1 \\ 0 & 2t \end{bmatrix}, \qquad Dq(p(x,y)) = \begin{bmatrix} 2(x^2-y) & 0 \\ 1 & 1 \\ 0 & 2(x+y^2) \end{bmatrix}$$

and so

$$Dh(x,y) = \begin{bmatrix} 2(x^2-y) & 0 \\ 1 & 1 \\ 0 & 2(x+y^2) \end{bmatrix} \begin{bmatrix} 2x & -1 \\ 1 & 2y \end{bmatrix} = \begin{bmatrix} 4x(x^2-y) & -2(x^2-y) \\ 2x+1 & -1+2y \\ 2(x+y^2) & 4y(x+y^2) \end{bmatrix}.$$

Common mistakes:

- Transposing the matrices. The derivative of a function $\mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix (rows correspond to output functions, columns correspond to input variables). The composition Dh is a function $\mathbb{R}^2 \to \mathbb{R}^3$, so Dh is a 3×2 matrix.
- Leaving s,t in the answer. Since h is a function of x and y but not s and t, only x and y should appear in the answer.

Problem #6 If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^4$, then the set of vectors orthogonal to both of them form a plane (i.e., a 2-dimensional space), so it would be arbitrary to choose just one and call it the cross product. For example, if $\mathbf{a} = (1,0,0,0)$ and $\mathbf{b} = (0,1,0,0)$, then should we choose $(0,0,1,0)$ or $(0,0,0,1)$? (Or $(0,0,3,-2)$ or $(0,0,\sqrt{2}/2,\sqrt{2}/2)$, all of which are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$?)

It certainly wouldn't be calculated the same way as the cross product in $\mathbb{R}^3$, since the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \end{bmatrix}$$

(where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}\}$ is the standard basis in $\mathbb{R}^4$) is not square and therefore does not have a well-defined determinant. On the other hand, this argument does not rule out the possibility that there might be some other way of calculating the cross product. It is possible to define the "cross product" of *three* vectors in $\mathbb{R}^4$; see problems 39–42 on p.62 of Colley.

Common mistakes (or rather, common misunderstandings about $\mathbb{R}^4$ and $\mathbb{R}^n$):

- *"The angle θ between* **a** *and* **b** *isn't well-defined."* Actually, it is; **a** and **b** together form a plane in which angles make sense. (This is true for any two vectors $\mathbf{a}, \mathbf{b}$ that live in the same $\mathbb{R}^n$.) We can even calculate the angle using the dot product: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$.
- *"Orthogonality does not make sense in $\mathbb{R}^3$."* Sure it does! Two vectors $\mathbf{a}, \mathbf{b}$ are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$. Note that the dot product, unlike the cross product, is well-defined in any dimension.
- *"$\mathbb{R}^4$ cannot be pictured, unlike $\mathbb{R}^3$."* Well, not all of it, but we can try to visualize *part* of it (to some extent that is what Math 223 is about). For example, the length of a vector, and the angle between two vectors, are still well-defined quantities that behave in expected ways (e.g., satisfying the triangle inequality, trigonometric identities, etc.)