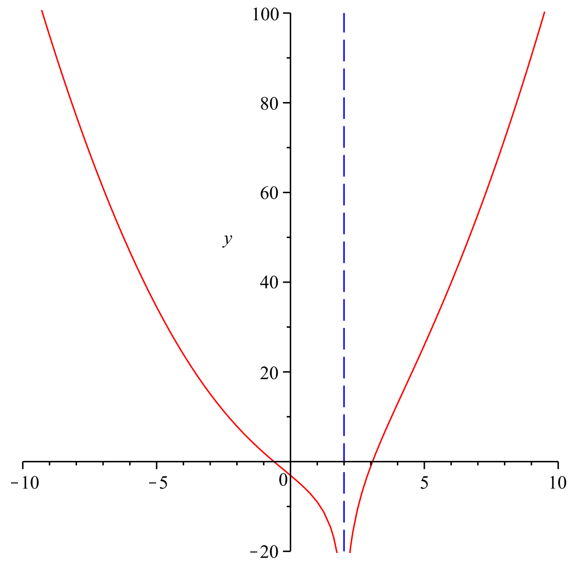
Math 141 Honors Problems #8 Comments

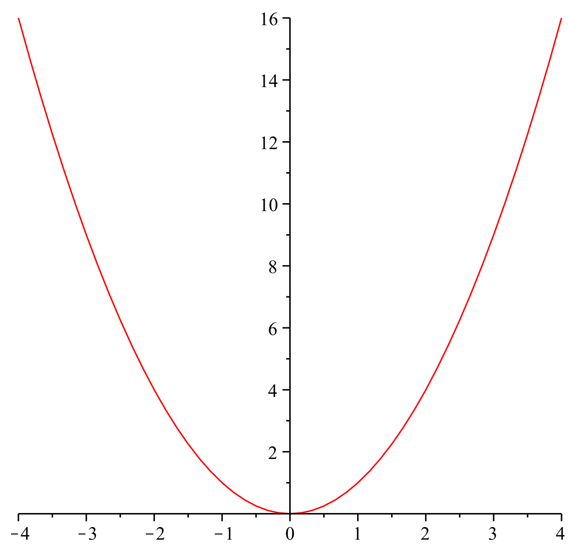
HP14: The function $b(x) = x^2 + \frac{\ln|x-2|}{1000}$ is continuous at all real numbers except at $x = 2$. The graph has a vertical asymptote, because

$$\lim_{x\to 2} b(x) \;=\; -\infty.$$

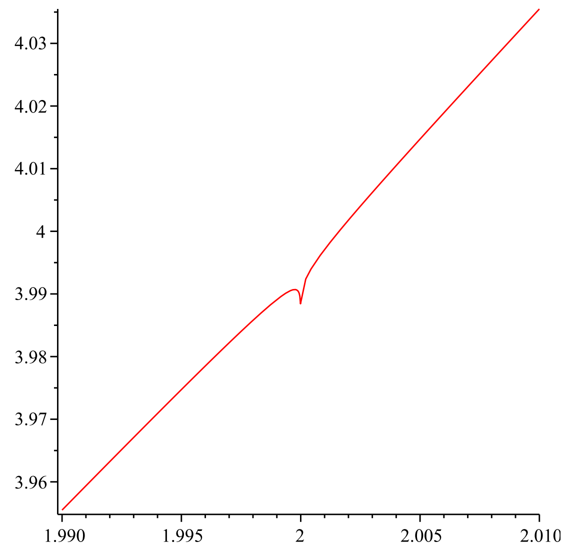
So the graph should look something like this:



However, your calculator will probably not be able to detect the asymptote, and will produce something like this:



Zooming in close may reveal the behavior, but it depends on your calculator. Here's what Maple produces for the interval $[-1.99, 2.01]$:



What's going on here is that you have to get **really** close to $x = 2$ in order to observe that $\lim_{x\to 2} b(x) = -\infty$. In many problems, particularly in calculus textbooks, it's enough to observe the value of $b(x)$ at x-values like 2 ± 0.1, 2 ± 0.01, 2 ± 0.001. Not here:

x	$b(x)$	x	$b(x)$
2.1	4.407697415	1.9	3.607697415
2.01	4.035494830	1.99	3.955494830
$2.001 = 2 + 10^{-3}$	3.997093245	1.999	3.989093245
2.00001	3.9885270746	1.99999	3.9884470746
$2 + 10^{-10}$	3.9769741495	$2 - 10^{-10}$	3.9769741487
$2 + 10^{-20}$	3.9539482981	$2 - 10^{-20}$	3.9539482981
$2 + 10^{-40}$	3.9078965963	$2 - 10^{-40}$	3.9078965963
$2 + 10^{-40}$	3.9078965963	$2 - 10^{-40}$	3.9078965963

It's no wonder that a calculator can't detect the asymptote, since numbers that are even this close to 2 have y-values close to 4.

Indeed, if r is large, then

$$b(2 \pm 10^{-r}) = 4 \pm 2 \cdot 10^{-r} + 10^{-2r} + \frac{\ln(10^{-r}}{1000} \approx 4 - \left(\frac{\ln 10}{1000}\right) r \approx 4 - 0.002303r.$$

In order to make this negative, we need r to be about 1737. To put this in perspective, even though the values of $b(x)$ decrease without bound as $x \to 2$, it is nevertheless the case that $b($2.0000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000001) > 0. That is *very* close to $x = 2$.

HP15: First, assume that the point P we are looking for is on the y-axis; call its coordinates $(0, y)$. Observe that $0 \leq y \leq h$. So we're trying to minimize the function

$$L(y) = d(P, X) + d(P, Y) + d(P, Z) = \sqrt{c^2 + y^2} + \sqrt{c^2 + y^2} + h - y = 2(c^2 + y^2)^{1/2} + h - y$$

on the interval $[0, h]$. First, we find the critical value(s) of L by setting $L'(y) = 0$:

$$\begin{aligned} L'(y) &= 2y(c^2 + y^2)^{-1/2} - 1 = 0 \\ 2y(c^2 + y^2)^{-1/2} &= 1 \\ 2y &= (c^2 + y^2)^{1/2} \\ 4y^2 &= c^2 + y^2 \\ y &= c/\sqrt{3}. \end{aligned}$$

Recall that the domain of L is $[0, h]$. Therefore, we have two cases:

Case I: $c/\sqrt{3} \leq h$. Then there are no critical values in the domain (except possibly at an endpoint if $c/\sqrt{3} = h$), so the minimum must be achieved at one of the endpoints, namely $y = 0$ or $y = h$. Note that

$$L(0) = 2c + h, \qquad L(h) = 2\sqrt{c^2 + h^2}.$$

Which of these is bigger? To figure this out, we'll reduce it to an algebraically simpler question, then ultimately use the very inequality that defines this case — namely, $c/sqrt3 > h$.

First, square both quantities:

$$L(0)^2 = 4c^2 + 4ch + h^2, \qquad L(h)^2 = 4c^2 + 4h^2.$$

Now, observe that

$$L(0)^2 - L(h)^2 = (4c^2 + 4ch + h^2) - (4c^2 + 4h^2) = 4ch - 3h^2 = h(4c - 3h).$$

But since $c/sqrt3 > h$, certainly $c > h$, so $4c - 3h > 0$. Therefore $L(0)^2 > L(h)^2$, which implies that $L(0) > L(h)$ (because both quantities are positive — they're sums of lengths). We conclude that the globla minimum of $L(y)$ occurs at $y = h$.

Case II: $c/sqrt3 < h$. Then the critical value $c/\sqrt{3}$ lies in the domain. Moreover,

$$\begin{aligned} L'(0) &= -1, \\ L'(h) &= \frac{2h}{\sqrt{c^2 + h^2}} - 1 \\ &> \frac{2h}{\sqrt{(h\sqrt{3})^2 + h^2)}} - 1 \\ &= \frac{2h}{\sqrt{4h^2}} - 1 = 0 \end{aligned}$$

so by the First Derivative Test, $y = c/\sqrt{3}$ is a local minimum, and in fact the global minimum.