Problem HP11: This ended up being too hard (the complete solution requires techniques we haven't discussed yet), but it will make a reappearance later in the semester.

Problem HP12: Many solvers gave the proof of the power rule for nonnegative integers, i.e.,

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{\left(\sum_{k=0}^{n}\binom{n}{k}x^k h^{n-k}\right) - x^n}{h}\\ &= \lim_{h\to 0}\frac{\sum_{k=0}^{n-1}\binom{n}{k}x^k h^{n-k}}{h}\\ &= \lim_{h\to 0}\sum_{k=0}^{n-1}\binom{n}{k}x^k h^{n-k-1}\\ &= \binom{n}{n-1}x^{n-1} \;=\; nx^{n-1}.\end{aligned}$$

The problem with this argument is that the binomial theorem does not hold for non-integer powers — indeed, neither the binomial coefficient $\binom{n}{k}$ is not well-defined unless k is a nonnegative integer, nor does it make sense to sum over all k from 0 to n.

The right approach is to rationalize the numerator in an expression such as

$$\frac{d}{dx}(x^{p/q}) = \lim_{h\to 0}\frac{(x+h)^{p/q} - x^{p/q}}{h}.$$

Recall that if $p/q = 1/2$, then the numerator of this expression is $(x+h)^{1/2} - x^{1/2} = \sqrt{x+h} - \sqrt{x}$, and we can rationalize the numerator by multiplying both numerator and denominator of the fraction by the conjugate expression is $(x+h)^{1/2} + x^{1/2} = \sqrt{x+h} + \sqrt{x}$.

For more general powers, we can use the algebraic identity

$$a^q - b^q = (a-b)(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + ab^{q-2} + b^{q-1})$$

where q is any positive integer. (This specializes to the well-known formulas for factoring differences of squares or cubes in the cases $q = 2$ and $q = 3$ respectively.) Putting $a = (x+h)^{p/q}$ and $b = x^{p/q}$, so that

$a^q = (x+h)^p$ and $b^q = x^p$, we see (assuming for the moment that $p \geq 0$ and $q > 0$) that

$$\begin{aligned}
\frac{d}{dx}(x^{p/q}) &= \lim_{h\to 0} \frac{(x+h)^{p/q} - x^{p/q}}{h} \\
&= \lim_{h\to 0} \frac{a-b}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h} \cdot \frac{a^q - b^q}{(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + ab^{q-2} + b^{q-1})} \right) \\
&= \lim_{h\to 0} \frac{(x+h)^p - x^p}{h\left((x+h)^{p(q-1)/q} + (x+h)^{p(q-2)/q}x^{p/q} + \cdots + (x+h)^{p/q}x^{p(q-2)/q} + x^{p(q-1)/q}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sum_{k=0}^{p} \binom{p}{k} x^k h^{p-k}\right) - x^p}{h\left((x+h)^{p(q-1)/q} + (x+h)^{p(q-2)/q}x^{p/q} + \cdots + (x+h)^{p/q}x^{p(q-2)/q} + x^{p(q-1)/q}\right)} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{p-1} \binom{p}{k} x^k h^{p-k}}{h\left((x+h)^{p(q-1)/q} + (x+h)^{p(q-2)/q}x^{p/q} + \cdots + (x+h)^{p/q}x^{p(q-2)/q} + x^{p(q-1)/q}\right)} \\
&= \lim_{h\to 0} \frac{h\sum_{k=0}^{p-1} \binom{p}{k} x^k h^{p-k-1}}{h\left((x+h)^{p(q-1)/q} + (x+h)^{p(q-2)/q}x^{p/q} + \cdots + (x+h)^{p/q}x^{p(q-2)/q} + x^{p(q-1)/q}\right)} \\
&= \lim_{h\to 0} \frac{\sum_{k=0}^{p-1} \binom{p}{k} x^k h^{p-k-1}}{(x+h)^{p(q-1)/q} + (x+h)^{p(q-2)/q}x^{p/q} + \cdots + (x+h)^{p/q}x^{p(q-2)/q} + x^{p(q-1)/q}} \qquad \text{(aha!)} \\
&= \frac{\binom{p}{p-1} x^{p-1}}{qx^{p(q-1)/q}}
\end{aligned}$$

because the denominator has q terms, all of which tend to $x^{p(q-1)/q}$ as $h \to 0$. This expression, meanwhile, becomes

$$\frac{p}{q}x^{p-1-(p(q-1)/q)} = \frac{p}{q}x^{(pq-q-pq+p)/q} = \frac{p}{q}x^{(p-q)/q} == \frac{p}{q}x^{p/q-1}$$

as desired.

For the case that $p < 0$ is an integer, one can use the definition of derivative as a limit to show that $d/dx(1/f(x)) = -f'(x)/f(x)^2$, then apply the positive case.