Problem HP9: There were two solutions I had in mind (of course, there are lots of others).

First, you could look for a function $p(x)$ whose second derivative has a *jump* discontinuity at $x = 0$. This would correspond to a cusp in the graph of $p'(x)$. The simplest function we know that has a cusp is probably the absolute value function. That function is piecewise linear, so $p(x)$ should be piecewise quadratic, and indeed

$$p(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x \leq 0 \end{cases}$$

has the desired property of being differentiable, but not second-differentiable.

More generally, let

$$q(x) = \begin{cases} x^n \text{ if } x \geq 0 \\ -x^n \text{ if } x \leq 0 \end{cases} .$$

Then $q(x)$ is certainly differentiable to all orders for $x \neq 0$. On the other hand, at $x = 0$, observe that

$$\lim_{x \to 0^+} \frac{d^n q}{dx^n} = n! \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{d^n q}{dx^n} = -n!$$

so $q(x)$ does not have a continuous n^{th} derivative.

Second, you could look for a function $p(x)$ whose second derivative has an *infinite* discontinuity at $x = 0$. Since we want $p'(x)$ to be continuous on $\mathbb{R}$, it had better have a vertical tangent line. This suggests that it should be something like $x^{1/3}$, and indeed

$$p(x) = x^{4/3}$$

is differentiable (because $p'(x) = \frac{4}{3}x^{1/3}$ is defined and continuous everywhere) but not second-differentiable (because $p''(x) = \frac{4}{9}x^{-2/3}$ is not defined at $x = 0$). More generally, let

$$q(x) = x^{n+1/3}.$$

Then dq/dx, d^2q/dx^2, $\ldots$, $d^{n-1}q/dx^{n-1}$ are power functions with positive powers, but $d^nq/dx^n = Cx^{-1/3}$ has an infinite discontinuity at $x = 0$. (Here C is some constant whose value we don't care about.)

Problem HP10a:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x + x^2} &= \lim_{x \to 0} \left(\frac{\sin x}{x} \frac{1}{1+x} \right) && \text{(algebra)} \\ &= \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \frac{1}{1+x} \right) && \text{(Limit Laws)} \\ &= 1 \cdot 1 \ = \boxed{1} \end{aligned}$$

Problem HP10b:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin^2(3x)}{x^2\cos x} &= \lim_{x\to 0}\left[\left(\frac{\sin(3x)}{x}\right)^2(\cos x)\right] && \text{(algebra)}\\
&= \left(\lim_{x\to 0}\frac{\sin(3x)}{x}\right)^2\left(\lim_{x\to 0}\cos x\right) && \text{(Limit Laws)}\\
&= \left(\lim_{x\to 0}\frac{\sin(3x)}{x}\right)^2\\
&= \left(\lim_{y\to 0}\frac{\sin y}{y/3}\right)^2 && \text{(algebra—substituting } y = 3x\text{)}\\
&= \left(3\lim_{y\to 0}\frac{\sin y}{y}\right)^2\\
&= 3^2 \;=\; \boxed{9}
\end{aligned}$$

Problem HP10c:

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\tan x}{\sin x} &= \lim_{x\to 0}\left(\frac{x}{\sin x}-\frac{\frac{\sin x}{\cos x}}{\sin x}\right) && \text{(algebra)}\\
&= \lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}-\frac{1}{\cos x}\right) && \text{(algebra)}\\
&= \left(\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}\right)-\left(\frac{1}{\lim_{x\to 0}\cos x}\right) && \text{(Limit Laws)}\\
&= 1-1 \;=\; \boxed{0}
\end{aligned}$$

Problem HP10d:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin^2\theta} &= \lim_{\theta\to 0}\frac{\cos\theta-1}{1-\cos^2\theta} && \text{(algebra)}\\
&= \lim_{\theta\to 0}\frac{-(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} && \text{(algebra)}\\
&= \lim_{\theta\to 0}\frac{-1}{1+\cos\theta} && \text{(algebra)}\\
&= \frac{-1}{1+1} \;=\; \boxed{\frac{-1}{2}}
\end{aligned}$$