Math 141 Honors Problems #13a
Due date: Tuesday, 12/1/09

HP24 [3 points] Let $f(x)$ and $g(x)$ be polynomials, and let a be a number greater than any of the zeroes of $g(x)$, so that the rational function $f(x)/g(x)$ is continuous on the interval (a, ∞). Prove that the improper integral

$$\int_a^\infty \frac{fx}{g(x)}\,dx$$

converges if and only if the degree of $g(x)$ is at least 2 more than the degree of $f(x)$. (Hint: Use the Comparison Theorem — see p. 429 — together with your knowledge about how the convergence or divergence of the integral $\int_1^\infty x^p\,dx$ depends on p.)

HP25 [6 points] The extremely important proof technique of *mathematical induction* is often used to show that some fact is true for every positive integer. For example, we've seen that the sums $S(n,p)$ defined by

$$S(n,p) = \sum_{i=1}^{n} i^p$$

have the closed-form formulas

$$S(n,1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \tag{1}$$

$$S(n,2) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

$$S(n,3) = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}. \tag{3}$$

Induction gives a way to prove that these formulas work for every positive integer n. Induction is vital in many other areas of mathematics, including sequences and series (which you'll see in Math 122/142) and identities involving binomial coefficients (think back to the first few days of class).

The technique of induction can be a bit confusing at first, but with a little practice, you can get used to how it works, and it is very much worth learning. A brief summary appears in the box on p.87 of the textbook, although it's not very enlightening by itself; you really have to work through several examples in your own to understand how the technique works. The Wikipedia article on induction (`http://en.wikipedia.org/wiki/Mathematical_induction`) has a more detailed description of induction, including a proof of the formula (1) (which is the "standard" example of induction). The formula (2) is proved inductively in Appendix F on p. A47. A slightly different example of induction appears on p. 90.

On the other hand, it is possible to prove formulas like (1), (2) and (3) without induction, as in Examples 4 and 5 on pp. A46–A47.

(1) Read all this material! Once you have done so, mimic the method of Example 5 to find a formula for $S(n,4)$, and check that it works for several values of n (say $1 \le n \le 5$). (If you want, you can check your formula for $S(n,4)$ against the Wikipedia article on Faulhaber's formula; see below.)
(2) Give another proof of your formula by induction.
(3) Finally, find a recursive formula for $S(n+1,p)$ in terms of $S(n,p)$, again by mimicking the method of Example 5. (Hint: In order to make the method work for all n, you will need binomial coefficients.)

For the curious, there is a general formula for $S(n, p)$ called *Faulhaber's formula* (see http://en.wikipedia.org/wiki/Faulhaber's_formula). However, the formula involves Bernoulli numbers, which are not easy to write in closed form; indeed, to give a general formula for them, you need mathematical induction again!