Math 141 Honors Problems #10
Due date: Tuesday, 11/3/09

HP16 [3 points] Consider a regular n-sided polygon whose radius is r. (That is, r is the distance from the center of the polygon to any one of the vertices.)

(i) Use geometry to find a formula for the area of this polygon in terms of r and n. (Hint: Break the polygon into a bunch of congruent isosceles triangles.) Call this area A.

(ii) Evaluate $\lim_{n\to\infty} A$. What does this mean geometrically?

HP17 [6 points – 2 per part] Let N be the population of the world, and let p be the probability that two randomly chosen people A and B have ever shaken hands. Observe that p is almost certainly not a constant, but rather a function of N, and in fact a *decreasing* function of N — every time a baby is born, there's one more person with whom most other people have not shaken hands. Indeed,

$$\lim_{N\to\infty} p(N) = 0.$$

Question: If you select a person at random, what is the probability that he or she has never shaken hands with anyone else?

Suppose we pick a person A at random. The probability that A has *not* shaken hands with any particular person B is $1 - p(N)$, and since there are $N - 1$ people in the world other than A, the probability that A has never shaken hands is

$$(1 - p(N))^{N-1}.$$

This expression can be unpleasant to evaluate, even for a calculator or computer, because N was estimated as 6,793,391,829 at the time I posted this problem (according to the US Census Bureau), and has undoubtedly increased since.

Fortunately, limits come to the rescue: for such a large value of N, the probability can be be estimated very closely by taking the limit as $N \to \infty$. (This limit will of course depend on $p(N)$, but it is easier to evaluate a limit than the 6,793,391,829th power of anything.) We can simplify the expression a little by replacing the exponent $N - 1$ with N, which doesn't affect the limit as $N \to \infty$ (you can convince yourself of this using the Limit Laws). So the answer to the question can be approximated as

$$L = \lim_{N\to\infty} (1 - p(N))^N.$$

As you'll see in this problem, the value of this limit depends on $p(N)$. That is, the probability that there is a handshake-free person somewhere depends on how fast $p(N)$ approaches zero as a function of N.

(i) Evaluate L if $p(N) = c/N$, where c is a positive constant. (Hint: This is actually a special case of one of the homework problems from §4.5.)

(ii) Evaluate L if $p(N) = c/N^2$.

(iii) Evaluate L if $p(N) = \dfrac{\ln N}{N}$.

Note: The handshaking model is an example of the very general and powerful idea of a *random graph*, which can be used to study many networks arising in nature. Other examples include the Internet; hydrogen bonding between water molecules in a block of ice; the spread of Dutch elm disease between trees in a forest; the most efficient way to locate wireless Internet routers and GPS satellites; Six Degrees of Kevin Bacon; and many others.

HP18 [3 points] Another limit that arises in the theory of random graphs is

$$\lim_{n\to\infty} n\left(1-\frac{c\ln n}{n}\right)^n,$$

where c is some positive real number.

Show that

$$\lim_{n\to\infty} n\left(1-\frac{c\ln n}{n}\right)^n = \begin{cases} \infty & \text{if } 0 < c < 1, \\ 1 & \text{if } c = 1, \\ 0 & \text{if } c > 1. \end{cases}$$