Sample Final of Math 121, Fall 2005

Print your **name** on **every page**. There are **7 pages** with **15 problems**. Two detachable blank pages are provided at the back of this test for use as a scratch paper only, and any work left on this scratch paper will NOT be graded.

($54 = 18 \times 3$ points) **Part I.** Multiple-choice problems. **Circle the correct answer**. **No partial credit** possible.

1. In three hours, the velocity of a car at each half hour was recorded as follows:

Time (hours)	0	.5	1	1.5	2	2.5	3
Velocity (mi/h)	25	45	50	55	40	45	37

What is the numerical estimate of the **average** velocity (in miles/hour) of the car over these three hours, obtained via the Simpson's approximation S_6?

(a) 130
(b) 133
(c) 136
(d) 137
(e) none of the above

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2. The slope of the tangent line to the curve $x^2 + x\ln y + xy^2 = 5 + \ln 2$ at the point $(1, 2)$ is

(a) $-\frac{3}{2}$
(b) $-\frac{4}{3}$
(c) $-\frac{4}{3} - \frac{2}{9}\ln 2$
(d) $-\frac{5}{6} - \frac{2}{9}\ln 2$
(e) none of the above

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3. If 600 cm^2 of material is available to make an open-top box with a square base, what is the largest possible volume of the box?

(a) $1000\sqrt{2}$
(b) $10\sqrt{2}$
(c) $10\sqrt{6}$
(d) $100\sqrt{6}$
(e) none of the above

4. Let f and g be functions with continuous derivatives f' and g', respectively, well defined on the real line, such that $\int_1^3 f(x)\,dx = 4$ and $\int_1^5 f(x)\,dx = 6$. Furthermore we have the following data.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-3	-1	2	-3
3	5	1	-1	-2
5	-5	0	-2	3

(i) $(g \circ f)'(3) =$
(a) -3
(b) 2
(c) -4
(d) 4
(e) none of the above

. . . .

(ii) If $h(x) = f(x)^2 g(x^2)$, then $h'(1) =$
(a) -15
(b) -48
(c) -42
(d) 36
(e) none of the above

. . . .

(iii) $\lim_{x\to 5} \dfrac{f(x)\,g(x)}{x^2 - 25} =$
(a) 1
(b) -1.5
(c) ∞
(d) -0.6
(e) none of the above

. . . .

(iv) If $H(x) = \int_0^{\ln x} f(t)^2\,dt$, then $H'(e^3) =$
(a) $\frac{25}{3}$
(b) $8e^3$
(c) 25
(d) $25e^{-3}$
(e) none of the above

. . . .

(v) $\displaystyle\int_0^{\ln 3} e^t f\left(e^t + 2\right) dt =$
(a) 4
(b) 0
(c) 2
(d) ∞
(e) none of the above

. . . .

(vi) $\displaystyle\int_1^3 x f'(x)\,dx =$
(a) 14
(b) 24
(c) 12
(d) 11
(e) none of the above

5. What is the exact volume of the solid obtained by revolving, about the y-axis, the region

$$R = \left\{(x,y) : 1 \le x \le \pi \text{ and } 0 \le y \le e^{x^2}\right\} \ ?$$

(a) $\pi e^{2\pi^2}$
(b) $e^{\pi^2} - e$
(c) $e^{\pi^2}\pi - e\pi$
(d) $2\pi e^{\pi^2} + 4\pi^2 e^{\pi^2}$
(e) none of the above

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6. What is the exact volume of the solid obtained by revolving, about the line $y = -1$, the region

$$R = \left\{(x,y) : 0 \le x \le 1 \text{ and } x \le y \le e^{3x}\right\} \ ?$$

(a) $\pi\left(\frac{1}{6}e^6 - \frac{1}{3}\right)$
(b) $\pi\left(\frac{2}{3}e^3 - \frac{5}{3}\right)$
(c) $\pi\left(\frac{4}{9}e^3 - \frac{4}{9}\right)$
(d) $\pi\left(\frac{1}{6}e^6 - \frac{1}{2}\right)$
(e) none of the above

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7. Consider the parametric curve γ defined by $(x,y) = \left(\frac{1}{3}t^3 - t + 3, t^2 + 1\right)$ with $t \ge 0$.
(i) The slope of the line tangent to the parametric curve γ at the point $\left(\frac{11}{3}, 5\right)$ is
(a) $\dfrac{4}{3}$
(b) $\dfrac{3}{5}$
(c) $\dfrac{15}{11}$
(d) 4
(e) none of the above

(ii) The length of the part of γ that joins the points $(3,1)$ and $(9,10)$ is
(a) $\dfrac{348}{5}$
(b) 12
(c) 15
(d) $2\sqrt{3}$
(e) none of the above

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8. A cable that weighs 10 lb per linear foot is used to lift 900 lb of coal up a mine-shaft. How much work, measured in ft-lb, is done in lifting the coal from 200 feet below the ground to 30 feet below the ground?
(a) 380 000
(b) 348 500
(c) 297 500
(d) 353 000
(e) none of the above

9. If $\int_{-3}^{2} f(x)\,dx = -1$, $\int_{-1}^{5} f(x)\,dx = 8$, and $\int_{-3}^{5} f(x)\,dx = 6$, find $\int_{-1}^{2} f(x)\,dx$.
(a) 2
(b) -1
(c) 3
(d) -2
(e) none of the above

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10. Let $F(x) = \int_{0}^{x^2} e^{-f(t)}dt$ for a differentiable function f on $(-\infty, \infty)$. Then $F'(x) =$
(a) $2xe^{-f(x)}f'(x)$
(b) $-2xe^{-f(x)}f'(x)$
(c) $2xe^{-f(x^2)}$
(d) $-2xe^{-f(x^2)}$
(e) none of the above

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11. A tank of the shape of a circular cone with its vertex pointing downward (and its top horizontal) is completely filled with water. Assume that the radius of its circular top is 1 m and its height (i.e. the distance from the vertex to the top) is 1 m.
(i) What is the work, measured in newton-meter (i.e. joule or kg·m^2/s^2), needed to pump all of the water out of this tank over its top? (Note that the density of water is 1000 kg/m^3 and the gravitational acceleration is 9.8 m/s^2.)
(a) $\frac{9800\pi}{12}$
(b) $\frac{9800\pi}{3}$
(c) 9800π
(d) $\frac{9800}{4}\pi$
(e) none of the above

. . . .

(ii) Let $h(t)$ be the water level (i.e. the distance from the vertex to the water surface) in the tank t seconds after we start to pump the water out of this tank at the constant rate of 0.1 m^3/s. How fast, in m/s, is the water level dropping at the moment when the water level is 0.4 m? (Give the correct rate of change in its absolute value, ignoring the ±-sign.)
(a) $\frac{0.2778}{\pi}$
(b) $\frac{0.625}{\pi}$
(c) $\frac{0.35}{\pi}$
(d) $\frac{0.001}{\pi}$
(e) none of the above

Part II. True-false Problems. Circle the **correct answer**, T (standing for 'True') or F (standing for 'False'). **No partial credit** possible.

($8 = 4 \times 2$ points) **12**. Determine whether each of the following statements is true or false.

(1) T F · · · · · $\int_1^9 f(x)\,dx = \int_1^3 f(x)\,dx - \int_9^3 f(x)\,dx$ for any continuous function f on $(-\infty, \infty)$.

(2) T F ····· $\int_2^1 f(x)\,dx < 0$ for any **positive** (i.e. $f(x) > 0$ for all x) continuous function f on $(-\infty, \infty)$.

(3) T F ····· $\lim_{h\to 0} \frac{1}{h}\int_a^{a+h} f(t^2)\,dt = 2af(a^2)$ for any continuous function f on $(-\infty, \infty)$ and any $a \in (-\infty, \infty)$.

(4) T F ····· $\frac{1}{6}\int_3^9 f(t)\,dt \le |f(3)| + |f(9)|$ for any continuous function f on $(-\infty, \infty)$.

($14 = 7 \times 2$ points) **13**. The graph of a continuous function f on the closed interval $[-6, 7]$ is shown in the following figure, where the arc is a semicircle. Let $g(x) = \displaystyle\int_{-2}^{x} f(t)\,dt$ for $-6 \le x \le 7$.

(1) T F ····· $g(-6) < 0$.

(2) T F ····· $g(1) = g(3) = 6 - \pi$.

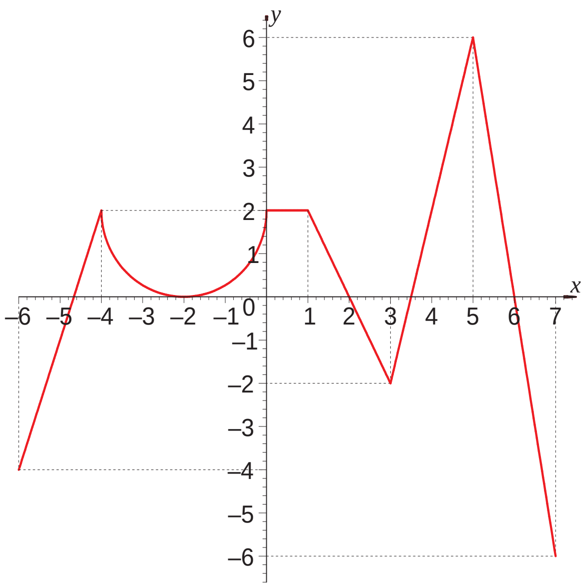
(3) T F ····· g is not differentiable at $x = -4,\ 0,\ 1,\ 3,\ 5$.

(4) T F ····· $g''(4) = 4$.

(5) T F ····· g has a local minimum at $x = \frac{7}{2}$.

(6) T F ····· g is concave up on the interval $(-4, 0)$.

(7) T F ····· g has an absolute maximum at $x = 2$ in the interval $[-6, 7]$.



Part III. Standard Essay Problems. **Show your work** to support your answers unless otherwise instructed. Solutions obtained only from calculators will not get any credit.

(10 points) **14**. Compute **EXACTLY ONE** of the following integrals and **CROSS OUT** the other one that is not to be graded.

(i) $\displaystyle\int_{-2}^{3} \frac{1}{(x-1)^8}\,dx.$

(ii) $\displaystyle\int \frac{x^2-5x+3}{x^3-9x}\,dx.$ (Note that $x^3-9x = x(x-3)(x+3)$.)

(14 points) **15**. The line $y = -x + 4$ and the parabola $y = x^2 - 2$ intersect at two point $(-3, 7)$ and $(2, 2)$, and bound (or enclose) a unique bounded region R. (i) Find the area A of the region R. (ii) Find the volume V, as a single concrete definite integral **without actually computing the value**, of the solid that has the region R (in the xy-plane) as its base such that each of its cross-sections perpendicular to the x-axis is a half-disk. (Note that this solid is **not** a solid of revolution.) (iii) Find the length ℓ of the whole boundary of the region R, as a single concrete definite integral **without actually computing the value**.

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