Math 141 Homework #10
Due Tuesday, 10/30/07
Extra Problems

Problem #1 Consider a regular n-sided polygon whose radius is r. (That is, r is the distance from the center of the polygon to any one of the vertices.)

(#1a) Use geometry to find a formula for the area of this polygon in terms of r and n. (Hint: Break the polygon into a bunch of congruent isosceles triangles.) Call this area A.

(#1b) Evaluate $\lim_{n\to\infty} A$. What geometric fact have you have just proven?

Problem #2 Let N be the population of the world, and let p be the probability that two randomly chosen people A and B have ever shaken hands. (Note that p is probably not a constant, but a function of N. If so, then $p = p(N)$ is almost certainly be a *decreasing* function of N.)

What is the probability that there is someone, somewhere, who has never shaken hands with anyone else?

The probability that any *particular* person has never shaken hands can be shown to be

$$(1 - p(N))^{N-1}.$$

This expression can be unpleasant to evaluate, particularly since N was estimated as 6,602,224,175 in July (according to the CIA). Fortunately, limits come to the rescue: for such a large value of N, the probability can be be estimated very closely by taking the limit as $N \to \infty$; call this limit Y. We can simplify the expression a little by replacing the exponent $N - 1$ with N (after all, misestimating the world's population by one person is not going to change the final answer). That is, the answer to the question can be approximated as

$$Y = \lim_{N\to\infty} (1 - p(N))^N.$$

(#2a) Evaluate Y if $p(N) = c/N$, where c is a positive constant. (Hint: This is actually a special case of one of the homework problems from §4.5.)

(#2b) Evaluate Y if $p(N) = c/N^2$.

(#2c) Evaluate Y if $p(N) = \dfrac{\ln N}{N}$.

Note: This handshaking model is an example of the very general and powerful idea of a *random graph*, which can be used to study many networks arising in nature. Other examples include the Internet; hydrogen bonding between water molecules in a block of ice; the spread of Dutch elm disease between trees in a forest; the most efficient way to locate wireless Internet routers and GPS satellites; Six Degrees of Kevin Bacon; and many others.

Bonus Problem Another limit that arises in the theory of random graphs is

$$\lim_{n\to\infty} n\left(1-\frac{c\ln n}{n}\right)^n,$$

where c is some positive real number.

Show that

$$\lim_{n\to\infty} n\left(1-\frac{c\ln n}{n}\right)^n = \begin{cases} \infty & \text{if } 0<c<1, \\ 1 & \text{if } c=1, \\ 0 & \text{if } c>1. \end{cases}$$