Math 141 Bonus Problems
Due Date: Tuesday, December 11, 4:00 PM

These are some of the tougher Bonus Problems culled from homework assignments. You can work on these problems at any point during the semester, and can turn them in at any time before the final exam for extra credit.

Problem #1 Let $T(x)$ be the function with domain $(0, \infty)$ defined as follows:

- $T(a/b) = 1/b$, if a/b is a fraction in lowest terms;
- $T(x) = 0$, if x is an irrational number.

So, e.g., $T(1/2) = 1/2$, $T(0.375) = 1/8$ (because $0.375 = 3/8$), $T(\pi) = 0$.

For which values of a is $T(x)$ continuous at a?

Problem #2 For all positive integers n, construct a function q such that q is $(n-1)^{th}$-order differentiable but not n^{th}-order differentiable. That is,

$$q, \quad \frac{dq}{dx}, \quad \frac{d^2q}{dx^2}, \quad \ldots, \quad \frac{d^{n-1}q}{dx^{n-1}}$$

are all defined and continuous, but $\dfrac{d^nq}{dx^n}$ has a discontinuity (again, say, at $x = 0$).

Problem #3 The Extreme Value Theorem states that if a function $f(x)$ is continuous on a closed interval I, then f achieves a global maximum and a global minimum on I. In class on Tuesday 10/16, we discussed the possibility that f has infinitely many critical numbers in I. Let's call a function *wild* if it exhibits this behavior, and *tame* otherwise. As we saw in class, an example of a wild function is

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1 \\ 0 \text{ if } x = 1 \end{cases}$$

on the interval $I - [0, 1]$.

We know that if f is tame, then we can find the global minimum and maximum of f on I by listing all the (finitely many) critical numbers and endpoints, evaluating f at each of them, and comparing the values. We'd like to prove that this method still works even for wild functions.

(#3a) First, explain why the range of f must be an interval. That is, rule out the possibility that the range is something like

$$[-4, 0) \cup (1, 3]$$

or

$$[-4, 0) \cup (0, 3)$$

or

$$\{1, 2, 3, 5, 8, 13, 21\}.$$

The next step is to figure out whether the interval is open, closed or half-and-half, and whether it is finite or infinite.

(#3b) Second, prove the following Lemma. If $\{x_1, x_2, x_3, \dots\}$ is an *infinite* set of numbers in I, then there is some number $a \in I$ that is an "accumulation point" of the x_i's — that is, with the property that any open[1] interval containing a also contains at least one of the x_i's.

This Lemma is a key tool for the rest of the problem. If you don't see how to prove the Lemma, that's okay; you can still do the rest of the problem by assuming that the Lemma is true.

(#3c) Now prove that f is *bounded* on I; that is, there are numbers A and B (for "above" and "below") such that

$$B \leq f(x) \leq A$$

for every $x \in I$. (Hint: Think about what would have to happen if f is *not* bounded, and use the Lemma.)

Another way of saying this result is that the range of f is a subset of the interval $[B, A]$, therefore a finite interval. We might as well assume that the interval is one of the following:

$$[B, A], \qquad (B, A], \qquad [B, A), \qquad (B, A).$$

(#3d) Rule out the possibility that the range is an open or half-open interval. (Hint: The numbers $(A + B)/2$, $(2A + B)/3$, $(3A + B)/4$, $\dots$ all lie in the range; use this together with the Lemma.)

Problem #4 A limit that arises in the theory of random graphs is

$$\lim_{n \to \infty} n \left(1 - \frac{c \ln n}{n}\right)^n,$$

where c is some positive real number.

Show that

$$\lim_{n \to \infty} n \left(1 - \frac{c \ln n}{n}\right)^n = \begin{cases} \infty & \text{if } 0 < c < 1, \\ 1 & \text{if } c = 1, \\ 0 & \text{if } c > 1. \end{cases}$$

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[1] Or possibly half-open, if a is an endpoint of I, but you can ignore that case if you want to.

Problem #5 The extremely important proof technique of *mathematical induction* is often used to show that some fact is true for every positive integer. For example, the sums

$$S(n,p) = \sum_{i=1}^{n} i^p$$

that come up in integration have closed-form formulas like

$$S(n,1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \tag{1}$$

$$S(n,2) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

$$S(n,3) = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}. \tag{3}$$

To prove that these formulas work for every positive integer n, you can use induction. Induction is vital in many other areas of mathematics, including sequences and series (which you'll see in Math 122/142) and identities involving binomial coefficients (think back to the first few days of class).

The technique of induction can be a bit confusing at first, but with a little practice, you can get used to how it works, and it is very much worth learning. A brief summary appears in the box on p.87 of the textbook, although it's not very enlightening by itself; you really have to work through a couple of examples to get the hang of how the technique works. The Wikipedia article on induction at

`http://en.wikipedia.org/wiki/Mathematical_induction`

has a more detailed description of induction that is readable and accurate, including a proof of the formula (1) (which is the "standard" example of induction). The formula (2) is proved inductively in Appendix F on p. A47. A slightly different example of induction appears on p. 90.

On the other hand, it is possible to prove formulas like (1), (2) and (3) without induction, as in Examples 4 and 5 on pp. A46–A47.

Read all this material. Once you have done so, mimic the method of Example 5 to find a formula for $S(n,4)$, and check that it works for several values of n (say $1 \leq n \leq 5$). (If you want, you can check your formula for $S(n,4)$ against the Wikipedia article on Faulhaber's formula; see below.)

Next, give another proof of your formula by induction.

Finally, find a recursive formula for $S(n+1,p)$ in terms of $S(n,p)$, again by mimicking the method of Example 5. (Hint: In order to make the method work for all n, you will need binomial coefficients.)

For the curious, there is a general formula for $S(n,p)$ called Faulhaber's formula; see

`http://en.wikipedia.org/wiki/Faulhaber's_formula`.

However, the formula involves things called Bernoulli numbers, which are not easy to write in closed form; indeed, to give a general formula for them, you need mathematical induction again!

Problem #6 Generalize the result of Problem #2 to cover functions that are algebraic but not rational (i.e., involve roots as well as polynomials; see p. 33).

Problem #7 Read the Wikipedia article on tabular integration at

`http://en.wikipedia.org/wiki/Integration_by_parts#Tabular_integration_by_parts`

and write a clear and correct explanation of how and why it works.

Problem #8 Suppose that you are given two polynomials $f(x), g(x)$ such that $g(x)$ has no nonnegative roots. How can you tell whether

$$\int_0^\infty \frac{f(x)}{g(x)}\,dx$$

converges or diverges? (Hint: Make an educated guess, then prove it using the Comparison Theorem on p. 429.)

Problem #9 Generalize the result of the previous problem to include functions that are algebraic but not rational (i.e., involve roots as well as polynomials; see p. 33 of the textbook).