# Complete Graphs

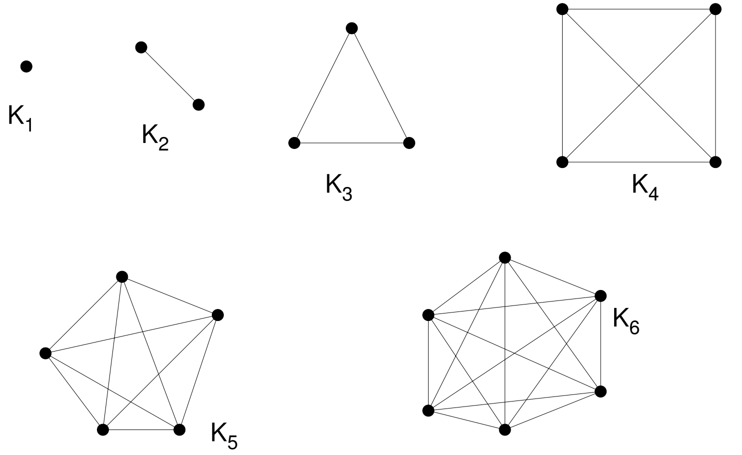
Let $N$ be a positive integer.

**Definition:** A **complete graph** is a graph with $N$ vertices and an edge between every two vertices.

- There are no loops.
- Every two vertices share exactly one edge.

We use the symbol $K_N$ for a complete graph with $N$ vertices.

# Complete Graphs



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**The number of edges in $K_N$ is $\dfrac{N(N-1)}{2}$.**

- This formula **also** counts the **number of pairwise comparisons** between $N$ candidates (recall §1.5).
- The Method of Pairwise Comparisons can be modeled by a complete graph.
  - Vertices represent candidates
  - Edges represent pairwise comparisons.
  - Each candidate is compared to each other candidate.
  - No candidate is compared to him/herself.

# Hamilton Circuits in $K_N$

**How many different Hamilton circuits does $K_N$ have?** ★

- Let's assume $N = 3$.
- We can represent a Hamilton circuit by listing all vertices of the graph in order.
- The first and last vertices in the list must be the same. All other vertices appear exactly once.
- We'll call a list like this an "itinerary".

Some possible itineraries:

A,C,D,B,A Y,X,W,U,V,Z,Y Q,W,E,R,T,Y,Q

# Hamilton Circuits in $K_N$

**How many different Hamilton circuits does $K_N$ have?**

- The first/last vertex is called the "reference vertex".
- Changing the reference vertex changes the itinerary but does not change the Hamilton circuit, because the same edges are traveled in the same directions.
- That is, different itineraries can correspond to the same Hamilton circuit.

# Hamilton Circuits in $K_N$

Changing the reference vertex does not change the Hamilton circuit.

For example, these itineraries all represent the same Hamilton circuit in $K_4$ (with edges AC, CD, DB, BA).

| | |
|---|---|
| A,C,D,B,A | (reference vertex: A) |
| B,A,C,D,B | (reference vertex: B) |
| D,B,A,C,D | (reference vertex: C) |
| C,D,B,A,C | (reference vertex: D) |

Every Hamilton circuit in $K_N$ can be described by exactly $N$ different itineraries (since there are $N$ possible reference vertices).

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- $\ldots$
- 2 possibilities for the $(N-1)$st vertex

# Hamilton Circuits in $K_N$

So, how many possible itineraries are there?

- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- ...
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the $N$th vertex

# Hamilton Circuits in $K_N$

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- $N$ possibilities for the reference vertex
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- $\ldots$
- 2 possibilities for the $(N - 1)$st vertex
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- and then the reference vertex again.

# Hamilton Circuits in $K_N$

So, how many possible itineraries are there?

- ~~$N$ possibilities for the reference vertex~~
- $N - 1$ possibilities for the next vertex
- $N - 2$ possibilities for the vertex after that
- ...
- 2 possibilities for the $(N - 1)$st vertex
- 1 possibility for the $N$th vertex
- and then the reference vertex again.

If we are counting Hamilton circuits, then we don't care about the reference vertex.

# Hamilton Circuits in $K_N$

Number of **itineraries**:

$$N \times (N-1) \times (N-2) \times \cdots \times 3 \times 2 \times 1 \quad = \quad \boxed{\textbf{N!}}$$

Number of **Hamilton circuits**:

$$(N-1) \times (N-2) \times \cdots \times 3 \times 2 \times 1 \quad = \quad \boxed{\textbf{(N-1)!}}$$

There are $N$ times as many itineraries as Hamilton circuits (because, again, every Hamilton circuit can be represented by $N$ different itineraries).

# Hamilton Circuits in $K_N$

For every $N \geq 3$,

**The number of Hamilton circuits in $\mathbf{K_N}$ is $\mathbf{(N-1)!}$.**

In comparison, for every $N \geq 1$,

**The number of edges in $\mathbf{K_N}$ is $\mathbf{\dfrac{N(N-1)}{2}}$.**

# Hamilton Circuits in $K_N$

| **Vertices** N | **Edges** $N(N-1)/2$ | **Hamilton circuits** $(N-1)!$ |
|---|---|---|
| 1 | 0 | |
| 2 | 1 | |
| 3 | 3 | 2 |
| 4 | 6 | 6 |
| 5 | 10 | 24 |
| 6 | 15 | 120 |
| 7 | 21 | 620 |
| ... | ... | ... |
| 16 | 120 | 1307674368000 |

# Hamilton Circuits in $K_3$

**Itineraries in $K_3$:**

| | |
|---|---|
| A,B,C,A | A,C,B,A |
| B,C,A,B | B,A,C,B |
| C,A,B,C | C,B,A,C |

# Hamilton Circuits in $K_3$

**Itineraries in $K_3$:**

| | |
|---|---|
| A,B,C,A | A,C,B,A |
| B,C,A,B | B,A,C,B |
| C,A,B,C | C,B,A,C |

- Each column of the table gives 3 itineraries for the same Hamilton circuit (with different reference vertices).
- The number of Hamilton circuits is $(3-1)! = 2! = 2$.

# Hamilton Circuits in $K_4$

**All itineraries in $K_4$:**

| | | | | | |
|---|---|---|---|---|---|
| ABCDA | ABDCA | ACBDA | ACDBA | ADBCA | ADCBA |
| BCDAB | BDCAB | BDACB | BACDB | BCADB | BADCB |
| CDABC | CABDC | CBDAC | CDBAC | CADBC | CBADC |
| DABCD | DCABD | DACBD | DBACD | DBCAD | DCBAD |

- Each column lists 4 itineraries for the same Hamilton circuit.
- The number of Hamilton circuits is $(4-1)! = 3! = 6$.

# Hamilton Circuits in $K_4$

**All itineraries in** $K_4$ (without repeating the reference vertex):

| | | | | | |
|---|---|---|---|---|---|
| ABCD | ABDC | ACBD | ACDB | ADBC | ADCB |
| BCDA | BDCA | BDAC | BACD | BCAD | BADC |
| CDAB | CABD | CBDA | CDBA | CADB | CBAD |
| DABC | DCAB | DACB | DBAC | DBCA | DCBA |

**Where have you seen this table before?**

# Hamilton Circuits in $K_4$

**All itineraries in** $K_4$ (without repeating the reference vertex):

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|---|---|---|---|---|---|
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| CDAB | CABD | CBDA | CDBA | CADB | CBAD |
| DABC | DCAB | DACB | DBAC | DBCA | DCBA |

**Where have you seen this table before?**

- It's the same as the list of sequential coalitions in a weighted voting system.
- That's another reason why the number of itineraries on $N$ vertices is $N!$.

# The Traveling Salesman Problem

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The **Traveling Salesman Problem (TSP)** is the problem of finding a **minimum-weight Hamilton circuit** in $K_N$. In other words, **what is the most efficient way to visit all vertices?**